TUTTE AND JONES POLYNOMIALS OF LINK FAMILIES

SLAVIK JABLAN, LJILJANA RADOVIĆ†, RADMILA SAZDANOVIĆ††,

The Mathematical Institute, Knez Mihailova 36,
P.O.Box 367, 11001 Belgrade,
Serbia
sjablan@gmail.com

Faculty of Mechanical Engineering†
A. Medvedeva 14
18 000 Niš
Serbia
ljradovic@gmail.com

The George Washington University††
Department of Mathematics,
Monroe Hall of Government,
2115 G Street, NW,
Washington, D.C. 20052,
USA
radmilas@gmail.edu

Abstract

This article contains general formulas for Tutte and Jones polynomials for families of knots and links given in Conway notation and "portraits of families"–plots of zeroes of their corresponding Jones polynomials.

1. Introduction

Knots and links (or shortly KLs) will be given in Conway notation [Con, Ro, Cau, JaSa].

Definition 1. For a link or knot L given in an unreduced* Conway notation $C(L)$ denote by S a set of numbers in the Conway symbol excluding numbers denoting basic polyhedron and zeros (determining the position of tangles in the vertices of polyhedron) and let $\tilde{S} = \{a_1, a_2, \ldots, a_k\}$ be a non-empty subset of S. Family $F_{\tilde{S}}(L)$ of knots or links derived from L consists of all knots or links L' whose Conway symbol is obtained by substituting all $a_i \neq \pm 1$, by $sgn(a_i)|a_i + k_{a_i}|$, $|a_i + k_{a_i}| > 1$, $k_{a_i} \in Z$. [JaSa].

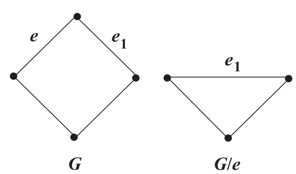
An infinite subset of a family is called *subfamily*. If all k_{a_i} are even integers, the number of components is preserved within the corresponding subfamilies, i.e., adding full-twists preserves the number of components inside the subfamilies.

*The Conway notation is called unreduced if in symbols of polyhedral links elementary tangles 1 in single vertices are not omitted.

Definition 2. A link given by Conway symbol containing only tangles ± 1 and ± 2 is called a *source link*.

A *graph* is defined as a pair (V, E), where V is the *vertex set* and $E \subseteq V \times V$ the *edge set*. We consider only *undirected* graphs, meaning (x, y) is the same as (y, x). A *loop* is an edge (x, x) between the same vertex, and a bridge is an edge whose removal disconnects two or more vertices (i.e. there is no longer a path between them) [ThoPe].

Two operations are essential to understanding the Tutte polynomial definition. These are: *edge deletion* denoted by $G - e$, and *edge contraction* G/e. The latter involves first deleting e, and then merging its endpoints as follows:



Definition 3. The *Tutte polynomial* of a graph $G(V, E)$ is a two-variable polynomial defined as follows:

$$T(G) = \begin{cases} 1 & E(\emptyset) & (1) \\ xT(G/e) & e \in E \text{ and } e \text{ is a bridge} & (2) \\ yT(G-e) & e \in E \text{ and } e \text{ is a loop} & (3) \\ T(G-e) + T(G/e) & e \in E \text{ and } e \text{ is neither a loop or a bridge} & (4) \end{cases}$$

The definition of a Tutte polynomial outlines a simple recursive procedure for computing it, but the order in which rules are applied is not fixed.

According to Thistlethwaite's Theorem, Jones polynomial of an alternating link, up to a factor, can be obtained from Tutte polynomial by replacements: $x \to -x$ and $y \to -\frac{1}{x}$ [Thi, Kau, Bo, ChaShro]. Moreover, from general formulas for Tutte polynomials with negative values of parameters we obtain Tutte polynomials expressed as Laurent polynomials. By the same replacements we obtain, up to a factor, Jones polynomials of non-alternating links.

A *cut-vertex* (or articulation vertex) of a connected graph is a vertex whose removal disconnects the graph [Char]. In general, a cut-vertex is a vertex of a graph whose removal increases the number of components [Har]. A *block* is a maximal biconnected subgraph of a given graph.

One-point union or *block sum* of two (disjoint) graphs G_1 and G_2, neither of which is a vertex graph, and which we shall denote as $G_1 * G_2$ is of particular interest. This one-point union is such that the intersection of G_1 and G_2 can only consist of a vertex[KuMu].

Decomposition of a graph G into a finite number of blocks G_1,...,G_n, denoted by

$$G = G_1 * G_2 * \ldots G_n$$

is called the *block sum* of G_1,...,G_n. The following formula holds for the Tutte polynomial of the block sum:

$$T(G_1 * G_2 * \ldots G_n) = T(G_1)T(G_2)\ldots T(G_n).$$

A dual graph $\overline{G}$ of a given planar graph G is a graph which has a vertex for each plane region of G, and an edge for each edge in G joining two neighboring regions, for a certain embedding of G. The Tutte polynomial of $\overline{G}$ can be obtained from $T(G)$ by replacements $x \to y$, $y \to x$, i.e. $T(\overline{G})(x,y) = T(G)(y,x)$.

There is a nice bijective correspondence between KLs and graphs: to obtain a graph from a projection of KL, first color every other region of the KL diagram black or white, so that the infinite outermost region is black. In the *checker-board coloring* (or *Tait coloring*) of the plane obtained, put a vertex at the center of each white region. Two vertices of a graph are connected by an edge if there was a crossing between corresponding regions in a KL diagram. In addition, to each edge of a graph we can assign the sign of its corresponding vertex of the KL diagram. Family of graphs corresponding to a family of link diagrams L will be denoted by $G(L)$.

Tutte polynomials were known for the following special families of graphs corresponding to the knots and links: circuit graphs C_p which correspond to the link family p, graphs corresponding to the link family $p\,2$, wheel graphs $Wh(n+1)$ corresponding to the family of antiprismatic basic polyhedra $(2n)^*$ (6^*, 8^*, 10^*, $\ldots$), so-called "hammock" graphs corresponding to the pretzel links $2,2,\ldots,2$ where tangle 2 occurs n times ($n \geq 3$), and graphs corresponding to the links of the form $(2n)^*: 2\,0 : 2\,0 \ldots : 2\,0$, where tangle 2 occurs n times ($6^*2\,0 : 2\,0 : 2\,0$, $8^*2\,0 : 2\,0 : 2\,0 : 2\,0$, $\ldots$) [ChaShro]. Recursive formulas for the computation of Tutte polynomials corresponding to the link families $3\,1\,3$, $3\,1\,2\,1\,3$, $3\,1\,2\,1\,2\,1\,3$, $3\,1\,2\,1\,2\,1\,2\,1\,3$, $\ldots$, $2\,1\,2$, $2\,1\,2\,1\,2$, $2\,1\,2\,1\,2\,1\,2$, $2\,1\,2\,1\,2\,1\,2\,1\,2$, $\ldots$, and the family of polyhedral links $.2:2$, $9^*2:2$, $\ldots$ are derived by F. Emmert-Streib [Emm]. Moreover, general formulas for the Jones polynomial are known for some other particular classes of knots and links, e.g., torus knots, or repeating chain knot with $3n$ crossings, proposed by L. Kauffman, given by Conway symbols of the form $3,2\,1\,1,2$; $(3,2)\,1,2\,1\,1,2$; $((3,2)\,1,2)\,1,2\,1\,1,2$; $(((3,2)\,1,2)\,1,2)\,1,2\,1\,1,2$; $((((3,2)\,1,2)\,1,2)\,1,2)\,1,2\,1\,1,2$; $\ldots$ [WuWa]. Zeros of the Jones polynomial are analyzed by X-S. Lin [Li], S. Chang and R. Shrock [ChaShro], F. Wu and J. Wang [WuWa], and X. Jin and F. Zhang [JiZha]. Experimental observations of many authors who have studied the distribution of roots of Jones polynomials for various families of knots and links are explained by A. Champanarekar and I. Kofman [ChaKo].

2. Tutte polynomials of KL families

2.1. *Family p*

The first family we consider is the family $\boldsymbol{p}$ ($p \geq 1$), which consists of the knots and links 1_1, 2_1^2, 3_1, 4_1^2, 5_1, $\ldots$ [ChaShro]. Graphs corresponding to links of this family are cycles of length p, which we can denote by $G(p)$. By deleting one edge

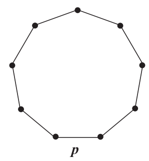


Figure 1: Cycle graph $G(p)$.

$G(p)$ gives the chain of edges of the length $p-1$ with the Tutte polynomial x^{p-1}, and by contraction it gives $G(p-1)$. Hence, $T(G(p))-T(G(p-1))=x^{p-1}$, and $T(G(1))=y$, so the general formula for the Tutte polynomial of the graph $G(p)$ is

$$T(G(p))=\frac{x^p-1}{x-1}+y-1.$$

2.2. *Family p q*

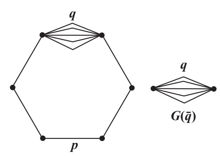


Figure 2: Graph $G(p\,q)$.

The link family ***p q*** gives the family of graphs, illustrated in Fig. 2, satisfying the relations

$$T(G(p\,q))-T(G((p-1)\,q))=x^{p-1}T(G(\overline{q})),$$

where $G(\overline{q})$ is the dual of the graph $G(q)$. Since the Tutte polynomial of the graph $G(0\,q)$ is $T(G(0\,q))=y^q$, the general formula for the Tutte polynomial of the graphs $G(p\,q)$ is

$$T(G(p\,q))=\frac{(x^p-1)(y^q-1)}{(x-1)(y-1)}+x^p+y^q-1.$$

2.3. *Family p 1 q*

The family of graphs (Fig. 3) corresponds to the link family ***p 1 q***. Since

$$T(G(p\,1\,q))-T(G(p\,1\,(q-1)))=x^{q-1}G(p+1),$$

the general formula for the Tutte polynomial of the graphs $G(p\,1\,q)$ is

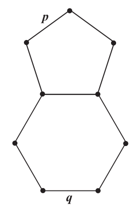


Figure 3: Graph $G(p\,1\,q)$.

$$T(G(p\,1\,q)) = \frac{x(x^p-1)(x^q-1)}{(x-1)^2} + \frac{(x^p+x^q+xy-x-y-1)y}{(x-1)}.$$

2.4. *Family* $\boldsymbol{p\,q\,r}$

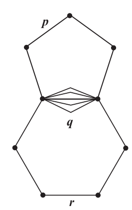


Figure 4: Graph $G(p\,q\,r)$.

The family of graphs (Fig. 4) corresponds to the family of rational links $\boldsymbol{p\,q\,r}$ and their Tutte polynomials satisfy the following relations

$$T(G(p\,q\,r)) - T(G((p-1)\,q\,r)) = x^{p-1}T(G(r\,q)).$$

The general formula for the Tutte polynomial of the graphs $G(p\,q\,r)$ is

$$T(G(p\,q\,r)) = \frac{(x+y)(x^p-1)(x^r-1)}{(x-1)^2} + \frac{y^q(x^{r+1}+x^p-x-1)}{(x-1)}$$
$$+\frac{(x^p-1)(x^r-1)(y^q-y^2)}{(x-1)^2(y-1)} - (x^r-y)y^q.$$

2.5. *Family* $\boldsymbol{p\;1\;1\;q}$

The graph of the link family $\boldsymbol{p\;1\;1\;q}$, illustrated in Fig. 5, resolves into the graph $G(p\,(q{+}1))$ and the block sum of the graphs $G(p{+}1)$ and $G(\overline{q})$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,1\,q)$ is

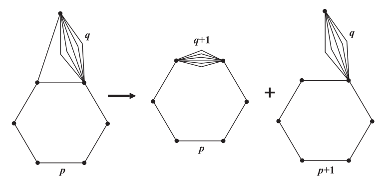


Figure 5: Graph $G(p\,1\,1\,q)$.

$$T(G(p\,1\,1\,q)) = T(G(p\,(q+1)) + (\frac{x^{p+1}-1}{x-1} + y - 1)(\frac{y^q-1}{y-1} + x - 1).$$

2.6. *Family p,q,r*

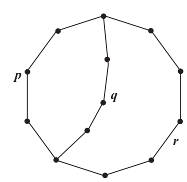


Figure 6: Graph $G(p, q, r)$.

The graph family from Fig. 6 corresponds to the family of pretzel links ***p,q,r***, whose Tutte polynomials satisfy the relations

$$T(G(p, q, r)) - T(G(p-1, q, r)) = x^{p-1}G(q+r)$$

The general formula for the Tutte polynomial of the graphs $G(p, q, r)$ is[†]

$$T(G(p, q, r)) = \frac{x^{p+q+r} + (x^{p+1} + x^{q+1} + x^{r+1})(y-1) - (x^p + x^q + x^r)y}{(x-1)^2} +$$

$$\frac{(xy - x - y)(xy - x - y - 1)}{(x-1)^2}.$$

2.7. *Antiprismatic basic polyhedra (2n)**

[†]In this paper we derived the general formula for Tutte polynomial of pretzel links with five parameters, which can be easily generalized to general formulas for pretzel links with an arbitrary number of parameters, by using the relation $T(G(p_1, \ldots, p_n)) = \frac{x^{p_n}-1}{x-1}T(G(p_1), \ldots, G(p_{n-1})) + T(G(p_1)) \ldots T(G(p_{n-1}))$ $(p \geq 3)$.

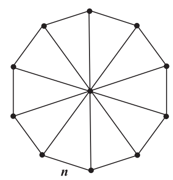


Figure 7: Wheel graph $Wh(n+1)$.

Basic polyhedron 6^* is the first member of the class of antiprismatic basic polyhedra ***(2n)***** $(n \geq 3)$: 6^*, 8^*, 10^*, ... The corresponding graphs are wheel graphs (Fig. 7) denoted by $Wh(n+1)$. Their Tutte polynomials are given by the general formula [ChaShro]:

$$T(G((2n)^*) = T(Wh(n+1)) = [\frac{1}{2}[(1+x+y)+[(1+x+y)^2-4xy]^{1/2}]]^n +$$

$$[\frac{1}{2}[(1+x+y)-[(1+x+y)^2-4xy]^{1/2}]]^n + xy - x - y - 1.$$

2.8. *Family .p*

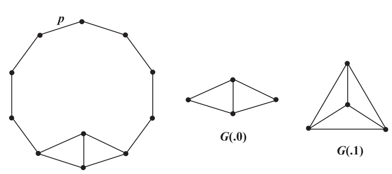


Figure 8: Graphs $G(.p)$.

The graphs from Fig. 8. correspond to the link family ***.p***. The Tutte polynomials of this graph family satisfy the relations

$$T(G(.p)) - T(G(.(p-1))) = x^{p-1}T(G(.0)),$$

$T(G(.0)) = x + 2x^2 + x^3 + y + 2xy + y^2$ and $T(G(.1)) = 2x + 3x^2 + x^3 + 2y + 4xy + 3y^2 + y^3$. The general formula for the Tutte polynomial of the graphs $G(.p)$ is

$$T(G(.p)) = \frac{x^p(2x^3+2x^2+y^2+2xy+y) - 2x^2 - 2x - y^2 - y - 2xy}{x-1}$$

$$-x^{p+1}(x+1) + y + 2xy + 2y^2 + y^3.$$

2.9. *Family p q 1 r*

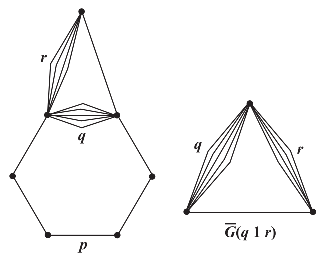


Figure 9: Graphs $G(p\,q\,1\,r)$ and $\overline{G}(q\,1\,r)$.

The graphs illustrated in Fig. 9 correspond to the link family ***p q 1 r***. In order to obtain formula for the Tutte polynomial we use the relations

$$T(G(p\,q\,1\,r)) - T(G((p-1)\,q\,1\,r)) = x^{p-1}T(\overline{G}(q\,1\,r)),$$

where $\overline{G}(q\,1\,r)$ denotes the graph from Fig. 9. Since its Tutte polynomial is

$$T(\overline{G}(q\,1\,r)) = (\frac{y^{r+1}-1}{y-1} + x - 1)\frac{y^q-1}{y-1} + x(\frac{y^r-1}{y-1} + x - 1),$$

for the Tutte polynomial of the graphs $G(p\,q\,1\,r)$ we obtain the general formula

$$T(G(p\,q\,1\,r)) = (\frac{y^{r+1}-1}{y-1}+x-1)\frac{y^q-1}{y-1}+x(\frac{y^r-1}{y-1}+x-1)\frac{x^p-1}{x-1}+y^q(\frac{y^{r+1}-1}{y-1}+x-1).$$

2.10. *Family p 1,q,r*

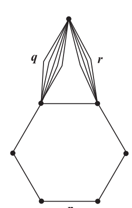


Figure 10: Graph $G(p\,1,q,r)$.

The graphs illustrated in Fig. 10 correspond to the link family ***p 1,q,r***. In order to obtain general formula for the Tutte polynomial we use the relations

$$T(G(p\,1,q,r))-T(G((p-1)\,1,q,r))=x^{p-1}T(\overline{G}(q\,1\,r)),$$

where $\overline{G}(q\,1\,r)$ denotes the graph from Fig. 9, which now occupies a different position with regards to the chain of edges p. The general formula for the Tutte polynomial of the graphs $G(p\,1,q,r)$ is

$$T(G(p\,1,q\,r))=(\frac{y^{r+1}-1}{y-1}+x-1)\frac{y^q-1}{y-1}+x(\frac{y^r-1}{y-1}+x-1)\frac{x^p-1}{x-1}+y(\frac{y^{q+r}-1}{y-1}+x-1).$$

2.11. ***Family p,q,r+***

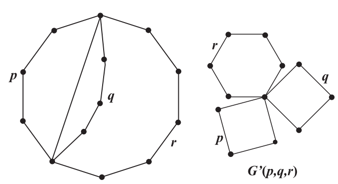


Figure 11: Graphs $G(p,q,r+)$ and $G'(p,q,r)$.

The Tutte polynomial of the graphs corresponding to the link family ***p,q,r+*** (Fig. 11) we obtain from the Tutte polynomial of the graphs $G(p,q,r)$, where the additional term is the Tutte polynomial of the graphs $G'(p,q,r)$. The general formula for the Tutte polynomial of the graphs $G(p,q,r+)$ is

$$T(G(p,q,r+))=T(G(p,q,r))+(\frac{x^p-1}{x-1}+y-1)(\frac{x^q-1}{x-1}+y-1)(\frac{x^r-1}{x-1}+y-1).$$

2.12. ***Family p 1 1 1 q***

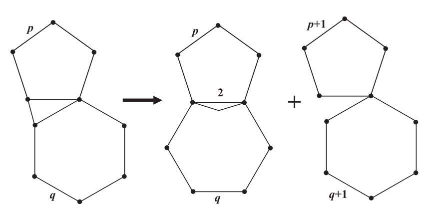


Figure 12: Resolving the graph $G(p\,1\,1\,1\,q)$ into graphs $G(p\,2\,q)$ and $G'(p+1,q+1)$.

By resolving the graph $G(p\,1\,1\,1\,q)$ into the graphs $G(p\,2\,q)$ and $G'(p+1,q+1)$ (Fig. 12) we obtain the general formula for the Tutte polynomial of the graphs $G(p\,1\,1\,1\,q)$

$$T(G(p\,1\,1\,1\,q)) = T(G(p\,2\,q)) + (\frac{x^{p+1}-1}{x-1}+y-1)(\frac{x^{q+1}-1}{x-1}+y-1).$$

2.13. *Family .p 1*

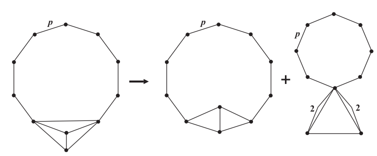


Figure 13: Resolving the graph $G(.p\,1)$ into the graph $G(.p)$ and the block sum of graphs $\overline{G}(2\,1\,2)$ and $G(p)$.

The graph of the link family ***.p 1*** (Fig. 13) resolves into the graph $G(.p)$ and the block sum of graphs $\overline{G}(2\,1\,2)$ and $G(p)$. The general formula for the Tutte polynomial of the graphs $G(.p\,1)$ is

$$T(G(.p\,1)) = T(G(.p)) + (x+x^2+y+2xy+2y^2+y^3)(\frac{x^p-1}{x-1}+y-1).$$

2.14. *Family .p:q*

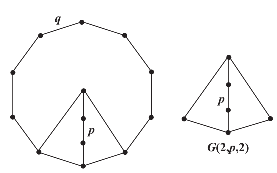


Figure 14: Graphs $G(.p:q)$ and $G(.p:0) = G(2,p,2)$.

The graph family illustrated in Fig. 14 corresponds to the link family ***.p:q***. In order to obtain general formula for the Tutte polynomial we can use the relations

$$T(G(.p:q)) - T(G(.p:(q-1))) = x^{q-1}T(G(2,p,2)).$$

where $G(.p:0)$ is the graph $G(2,p,2)$ with the Tutte polynomial

$$T(G(.p:0)) = T(G(2,p,2)) = \frac{x^p(x^3+x^2+x+y)+(2x+y+1)(xy-x-y)}{x-1}.$$

The general formula for the Tutte polynomial of the graph $G(.p:q)$ is

$$T(G(.p:q)) = \frac{x^q-1}{(x-1)^2}(x^p(x^3+x^2+x+y)+(2x+y+1)(xy-x-y))+$$
$$\frac{x^p-1}{x-1}(x+y)^2+x+y+y^2+y^3.$$

2.15. *Family .p.q*

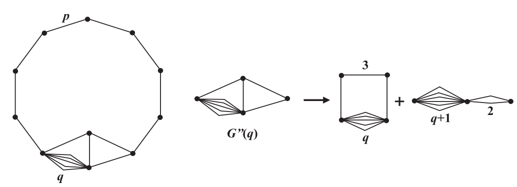


Figure 15: Graph $G(.p.q)$ and resolving of the graph $G''(q)$.

The graph family illustrated in Fig. 15 corresponds to the link family ***.p.q***. In order to obtain formula for the Tutte polynomial we can use the relations

$$T(G(.p.q)) - T(G(.(p-1).q)) = x^{p-1}T(G''(q)).$$

where $G''(q)$ is the graph which resolves into the graph $G(3\,q)$ and the block sum of graphs $G(\overline{q+1})$ and $G(\overline{2})$. The general formula for the Tutte polynomial of the graph $G''(q)$ is

$$T(G''(q)) = \frac{(1+2x+x^2+y)(xy-x-y)+y^q(x+x^2+y+xy+y^2)}{y-1},$$

and the Tutte polynomial of the graph $G(.p.q)$ is

$$T(G(.p.q)) = \frac{x^p-1}{x-1}T(G''(q)) + 2y\frac{y^{q+2}-1}{y-1} - y - y^{q+2}+$$
$$xy\frac{y^q-1}{y-1}+xy+x+x^2.$$

2.16. *Family .p:q 0*

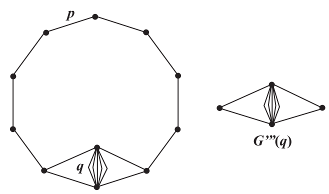


Figure 16: Graphs $G(.p:q\,0)$ and $G'''(q)$.

The graph family $G(.p:q\,0)$ illustrated in Fig. 16 corresponds to the link family ***.p:q 0***. In order to obtain general formula for the Tutte polynomial we can use the relations

$$T(G(.p.q))-T(G(.(p-1).q))=x^{p-1}T(G'''(q)).$$

The Tutte polynomial of the graph $G'''(q)$ is

$$T(G'''(q))=\frac{y^q-1}{y-1}(x+y)^2+x+x^2+x^3+y,$$

and the general formula for the Tutte polynomial of the graph $G(.p:q\,0)$ is

$$T(G(.p:q\,0))=\frac{x^p-1}{x-1}T(G'''(q))+2y\frac{y^{q+2}-1}{y-1}-2y-y^{q+2}-xy^q+$$

$$xy\frac{y^q-1}{y-1}+2xy+x+x^2+\frac{y^{q+1}-1}{y-1}-1.$$

2.17. *Family .p.q 0*

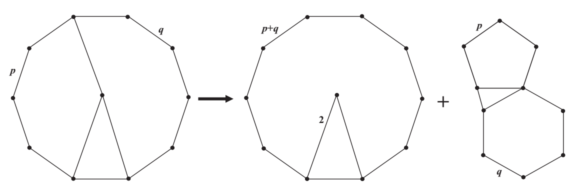


Figure 17: Resolving the graph $G(.p.q\,0)$ into the graphs $G((p+q)\,1\,2)$ and $G(p\,1\,1\,1\,q)$.

The graph family $G(.p.q\,0)$ illustrated in Fig. 16 corresponds to the link family ***.p.q 0***. The graph $G(.p.q\,0)$ resolves into the graphs $G((p+q)\,1\,2)$ and $G(p\,1\,1\,1\,q)$. The general formula for the Tutte polynomial of the graphs $G(.p.q\,0)$ is

$$T(G(.p.q\,0)) = T(G((p+q)\,1\,2)) + T(G(p\,1\,1\,1\,q)).$$

2.18. *Family p 1,q 1,r*

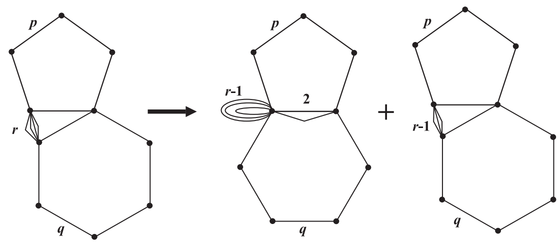


Figure 18: Resolving the graph $G(p\,1, q\,1, r)$.

The graph family $G(p\,1, q\,1, r)$ illustrated in Fig. 18 corresponds to the link family ***p 1,q 1,r***. In order to obtain general formula for the Tutte polynomial we can use the relations

$$T(G(p\,1, q\,1, r)) - T(G(p\,1, q\,1, (r-1))) = y^{r-1} T(G(p\,2\,q)).$$

The general formula for the Tutte polynomial of the graphs $G(p\,1, q\,1, r)$ is

$$T(G(p\,1, q\,1, r)) = T(G(p\,2\,q))\frac{y^r - 1}{y-1} +$$

$$\left(\frac{x^{p+1}-1}{x-1} + y - 1\right)\left(\frac{x^{q+1}-1}{x-1} + y - 1\right).$$

2.19. *Family p 1,q,r+*

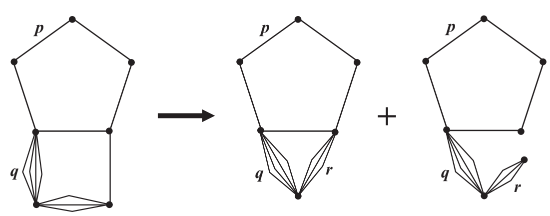


Figure 19: Resolving the graph $G(p\,1, q, r+)$.

The graph family $G(p\,1,q,r+)$ illustrated in Fig. 19 corresponds to the link family ***p 1,q,r+***. The graph $G(p\,1,q,r+)$ resolves into the graph $G(p\,1,q,r)$ and the block sum of the graphs $G(p+1)$, $G(\overline{q})$, and $G(\overline{r})$. The general formula for the Tutte polynomial of the graphs $G(p\,1,q,r+)$ is

$$T(G(p\,1,q,r+)) = T(G(p\,1,q,r)) + (\frac{x^{p+1}-1}{x-1}+y-1)(\frac{y^q-1}{y-1}+x-1)(\frac{y^r-1}{y-1}+x-1).$$

2.20. *Family p 1 1,q,r*

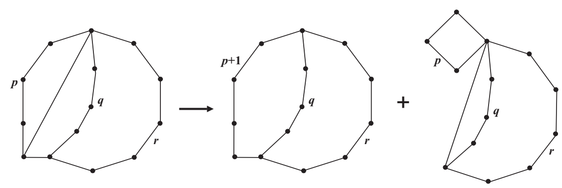


Figure 20: Resolving the graph $G(p\,1\,1,q,r)$.

The graph family $G(p\,1\,1,q,r)$ illustrated in Fig. 20 corresponds to the link family ***p 1 1,q,r***. The graph $G(p\,1\,1,q,r)$ resolves into the graphs $G((p+1),q,r)$ and the block sum of the graphs $G(1,q,r)$ and $G(p)$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,1,q,r)$ is

$$T(G(p\,1\,1,q,r)) = T(G((p+1),q,r)) + T(G(1,q,r))T(G(p)).$$

2.21. *Family p 1 q 1 r*

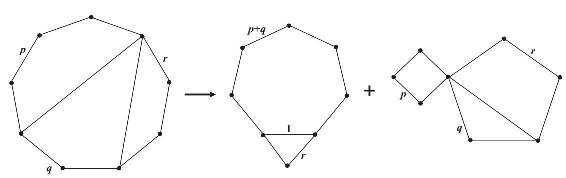


Figure 21: Resolving the graph $G(p\,1\,q\,1\,r)$.

The graph family $G(p\,1\,q\,1\,r)$ illustrated in Fig. 21 corresponds to the link family ***p 1 q 1 r***. The graph $G(p\,1\,q\,1\,r)$ resolves into the graph $G((p+q)\,1\,r)$ and the block sum of the graphs $G(q\,1\,r)$ and $G(p)$. The general formula for the Tutte polynomial of the graph $G(p\,1\,q\,1\,r)$ is

$$T(G(p\,1\,q\,1\,r)) = T(G((p+q)\,1\,r)) + T(G(q\,1\,r))T(G(p)).$$

2.22. *Family p,q,r,s*

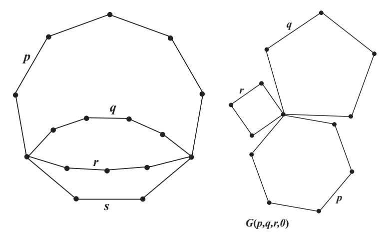


Figure 22: Graphs $G(p,q,r,s)$ and $G(p,q,r,0)$.

The graph family $G(p,q,r,s)$ illustrated in Fig. 22 corresponds to the link family ***p,q,r,s***. In order to obtain general formula for the Tutte polynomial we can use the relations

$$T(G(p,q,r,s)) - T(G(p,q,r,(s-1))) = x^{s-1}T(G(p,q,r)).$$

Since the Tutte polynomial of the graph $G(p,q,r,0)$ is

$$T(G(p,q,r,0)) = (\frac{x^p-1}{x-1}+y-1)(\frac{x^q-1}{x-1}+y-1)(\frac{x^r-1}{x-1}+y-1),$$

the general formula for the Tutte polynomial of the graphs $G(p,q,r,s)$ is

$$T(G(p,q,r,s)) = \frac{x^s-1}{x-1}T(G(p,q,r))+$$

$$(\frac{x^p-1}{x-1}+y-1)(\frac{x^q-1}{x-1}+y-1)(\frac{x^r-1}{x-1}+y-1).$$

2.23. *Family p 1 1 1 1 q*

The graph family $G(p\,1\,1\,1\,1\,q)$ illustrated in Fig. 23 corresponds to the link family ***p 1 1 1 1 q***. The graph $G(p\,1\,1\,1\,1\,q)$ resolves into the graph $G((p+1)\,1\,1\,q)$ and the block sum of the graphs $\overline{G}(q\,1\,2)$ and $G(p)$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,1\,1\,1\,q)$ is

$$T(G(p\,1\,1\,1\,1\,q)) = T(G((p+1)\,1\,1\,q)) + T(\overline{G}(q\,1\,2))T(G(p)).$$

2.24. *Family p q 1 1 r*

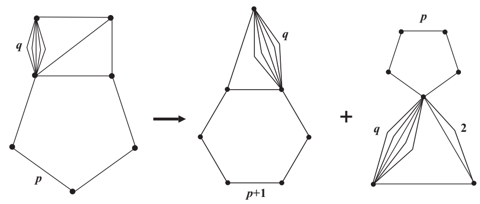


Figure 23: Resolving the graph $G(p\,1\,1\,1\,1\,q)$.

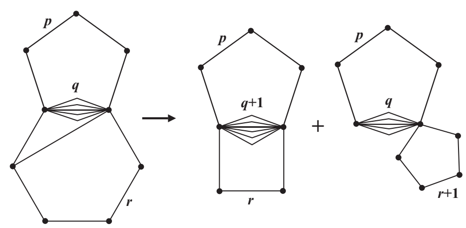


Figure 24: Resolving the graph $G(p\,q\,1\,1\,r)$.

The graph family $G(p\,q\,1\,1\,r)$ illustrated in Fig. 24 corresponds to the link family ***p q 1 1 r***. The graph $G(p\,q\,1\,1\,r)$ resolves into the graph $G(p\,(q+1)\,r)$ and the block sum of the graphs $G(p\,q)$ and $G(r+1)$. The general formula for the Tutte polynomial of the graphs $G(p\,q\,1\,1\,r)$ is

$$T(G(p\,q\,1\,1\,r)) = T(G(p\,(q+1)\,r)) + T(G(p\,q))T(G(r+1)).$$

2.25. *Family p q r s*

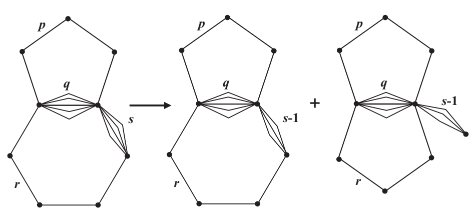


Figure 25: The relations for the graph $G(p\,q\,r\,s)$.

The graph family $G(p\,q\,r\,s)$ illustrated in Fig. 25 corresponds to the link family ***p q r s***. In order to obtain general formula for the Tutte polynomial we can use the relations

$$T(G(p\,q\,r\,s)) - T(G(p\,q\,r\,(s-1))) = y^{s-1}T(G(p\,q\,r)).$$

The general formula for the Tutte polynomial of the graphs $G(p\,q\,r\,s)$ is

$$T(G(p\,q\,r\,s)) = \frac{y^s-1}{y-1}T(G(p\,q\,r)) + x^r(\frac{x^{p+1}-1}{x-1}+y-1)+$$

$$x^r y\frac{y^{q-1}-1}{y-1}(\frac{x^p-1}{x-1}+y-1).$$

2.26. *Family p q,r,s*

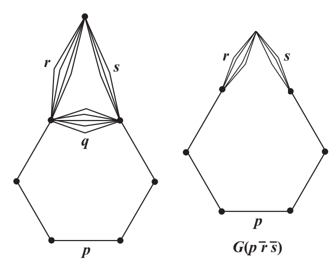


Figure 26: Graphs $G(p\,q,r,s)$ and $G(p\,\overline{r}\,\overline{s})$.

The graph family $G(p\,q,r,s)$ illustrated in Fig. 26 corresponds to the link family ***p q,r,s***. In order to obtain general formula for the Tutte polynomial we can use the relations

$$T(G(p\,q,r,s)) - T(G(p\,(q-1),r,s)) = y^{q-1}T(G(p))T(\overline{G}(r+s)).$$

After computing the Tutte polynomial of the graphs $G(p\,\overline{r}\,\overline{s})$

$$T(G(p\,\overline{r}\,\overline{s})) = \frac{x^p-1}{x-1}(\frac{y^r-1}{y-1}+x-1)(\frac{y^s-1}{y-1}+x-1)+(\frac{y^{r+s}-1}{y-1}+x-1),$$

we conclude that the general formula for the Tutte polynomial of the graphs $G(p\,q,r,s)$ is

$$T(G(p\,q,r,s)) = T(G(p\,\overline{r}\,\overline{s})) + \frac{y^q-1}{y-1}(\frac{x^p-1}{x-1}+y-1)(\frac{y^{r+s}-1}{y-1}+x-1).$$

2.27. *Family p,q,r+s*

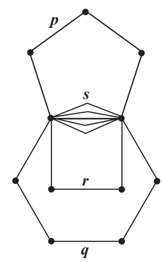


Figure 27: Graph $G(p, q, r + s)$.

The graph family $G(p, q, r + s)$ illustrated in Fig. 27 corresponds to the link family ***p,q,r+s***. In order to obtain general formula for the Tutte polynomial we can use the relations

$$T(G(p, q, r + s)) - T(G(p, q, r + (s - 1))) = y^{s-1}T(G(p))T(G(q))T(G(r))$$

and $T(G(p, q, r + 0)) = T(G(p, q, r))$. The general formula for the Tutte polynomial of the graphs $G(p, q, r + s)$ is

$$T(p, q, r+s) = \frac{y^s - 1}{y - 1}(\frac{x^p - 1}{x - 1}+y-1)(\frac{x^q - 1}{x - 1}+y-1)(\frac{x^r - 1}{x - 1}+y-1)+T(G(p, q, r)).$$

2.28. *Family (p,q) (r,s)*

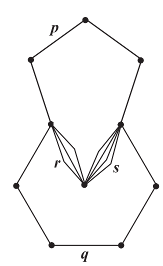


Figure 28: Graph $G(p, q)\,(r, s)$.

The graph family $G(p, q)\,(r, s)$ illustrated in Fig. 28 corresponds to the link family ***(p,q) (r,s)***. In order to obtain general formula for the Tutte polynomial we can use the relations

$$T(G((p, q)\,(r, s))) - T(G((p, (q - 1))\,(r, s))) = x^{q-1}T(G(p\,\overline{r}\,\overline{s})).$$

The general formula for the Tutte polynomial of the graphs $G(p,q)\,(r,s)$ is

$$T(G((p,q)\,(r,s))) = \frac{x^q-1}{x-1}T(G(p\,\overline{r}\,\overline{s})) + (\frac{x^p-1}{x-1}+y-1)(\frac{y^{r+s}-1}{y-1}+x-1).$$

2.29. *Family p q 1 r s*

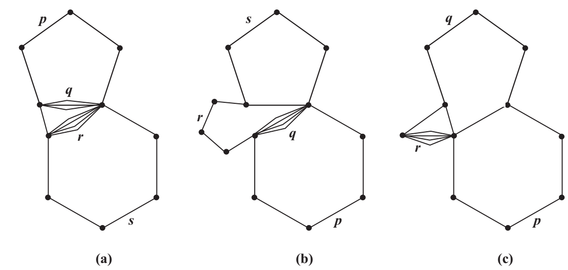


Figure 29: Graphs (a) $G(p\,q\,1\,r\,s)$; (b) $G(p\,q\,r\,1\,s)$; (c) $G(p\,1\,q\,1\,1\,r)$.

Graph $G(p\,q\,1\,r\,s)$ (Fig. 29a) resolves into the graph $G(p\,(q+r)\,s)$ and the block sum of graphs $G(p\,q)$ and $G(r\,s)$. The general formula for the Tutte polynomial of the graphs $G(p\,q\,1\,r\,s)$ is

$$T(G(p\,q\,1\,r\,s)) = T(G(p\,(q+r)\,s)) + T(G(p\,q))T(G(s\,r)).$$

2.30. *Family p q r 1 s*

Graph $G(p\,q\,r\,1\,s)$ (Fig. 29b) resolves into the graph $G(p\,q\,(r+s))$ and the block sum of graphs $G(p\,q\,r)$ and $G(s)$. The general formula for the Tutte polynomial of the graphs $G(p\,q\,r\,1\,s)$ is

$$T(G(p\,q\,r\,1\,s)) = T(G(p\,q\,(r+s))) + T(G(p\,q\,r))T(G(s)).$$

2.31. *Family p 1 q 1 1 r*

Graph $G(p\,1\,q\,1\,1\,r)$ (Fig. 29c) resolves into the graph $G((p+q)\,1\,1\,r)$ and the block sum of graphs $G(q\,1\,1\,r)$ and $G(p)$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,q\,1\,1\,r)$ is

$$T(G(p\,1\,q\,1\,1\,r)) = T(G((p+q)\,1\,1\,r)) + T(G(q\,1\,1\,r))T(G(p)).$$

2.32. *Family p q 1 1 1 r*

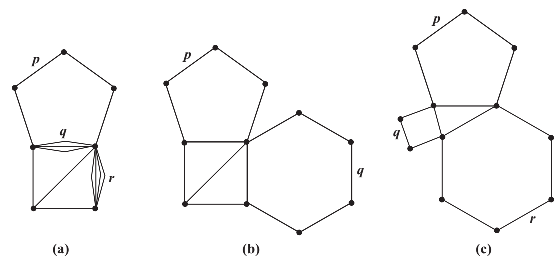


Figure 30: Graphs (a) $G(p\,q\,1\,1\,1\,r)$; (b) $G(p\,1\,1\,1\,1\,1\,q)$; (c) $G(p\,1, q\,1, r\,1)$.

Graph $G(p\,q\,1\,1\,1\,r)$ (Fig. 30a) resolves into the graph $G(p\,(q+1)\,1\,r)$ and the block sum of graphs $G(p\,q)$ and $G(2\,r)$. The general formula for the Tutte polynomial of the graphs $G(p\,q\,1\,1\,1\,r)$ is

$$T(G(p\,q\,1\,1\,1\,r)) = T(G(p\,(q+1)\,1\,r)) + T(G(p\,q))T(G(2\,r)).$$

2.33. *Family p 1 1 1 1 1 q*

Graph $G(p\,1\,1\,1\,1\,1\,q)$ (Fig. 30b) resolves into the graph $G(p\,1\,1\,1\,(q+1))$ and the block sum of graphs $G(p\,1\,1\,2)$ and $G(q)$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,1\,1\,1\,1\,q)$ is

$$T(G(p\,1\,1\,1\,1\,1\,q)) = T(G(p\,1\,1\,1\,(q+1))) + T(G(p\,1\,1\,2))T(G(q)).$$

2.34. *Family p 1,q 1,r 1*

Graph $G(p\,1, q\,1, r\,1)$ (Fig. 30c) resolves into the graph $G(p\,1\,q\,1\,r)$ and the block sum of graphs $G(p\,2\,r)$ and $G(q)$. The general formula for the Tutte polynomial of the graphs $G(p\,1, q\,1, r\,1)$ is

$$T(G(p\,1, q\,1, r\,1)) = T(G(p\,1\,q\,1\,r)) + T(G(p\,2\,r))T(G(q)).$$

2.35. *Family p 1 q,r,s*

Graph $G(p\,1\,q, r, s)$ (Fig. 31a) resolves into the graph $G((p+q), r, s)$ and the block sum of graphs $G(q, r, s)$ and $G(p)$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,q, r, s)$ is

$$T(G(p\,1\,q, r, s)) = T(G((p+q), r, s)) + T(G(q, r, s))T(G(p)).$$

2.36. *Family p q 1,r,s*

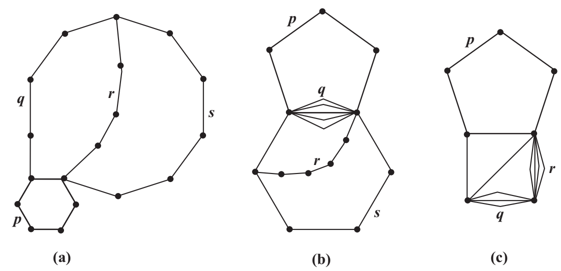


Figure 31: Graphs (a) $G(p\,1\,q, r, s)$; (b) $G(p\,q\,1, r, s)$; (c) $G(p\,1\,1\,1, q, r)$.

Graph $G(p\,q\,1, r, s)$ (Fig. 31b) resolves into the graph $G(p, r, s+q)$ and the block sum of graphs $G(p\,q)$ and $G(r+s)$. The general formula for the Tutte polynomial of the graphs $G(p\,q\,1, r, s)$ is

$$T(G(p\,q\,1, r, s)) = T(G(p, r, s+q)) + T(G(p\,q))T(G(r+s)).$$

2.37. *Family p 1 1 1,q,r*

Graph $G(p\,1\,1\,1, q, r)$ (Fig. 31c) resolves into the graph $G(p\,2, q, r)$ and the block sum of graphs $\overline{G}(q\,1\,r)$ and $G(p+1)$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,1\,1, q, r)$ is

$$T(G(p\,1\,1\,1, q, r)) = T(G(p\,2, q, r)) + T(\overline{G}(q\,1\,r))T(G(p+1)).$$

2.38. *Family p 1,q,r,s*

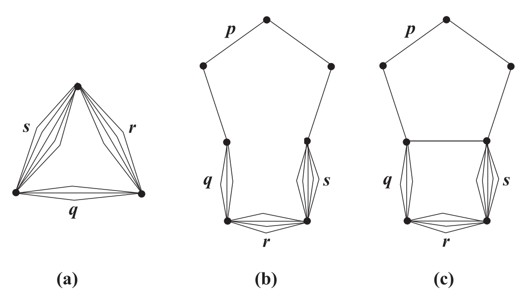


Figure 32: Graphs (a) $\overline{G}(q\,r\,s)$; (b) $G(p\,\overline{q}\,\overline{r}\,\overline{s})$; (c) $G(p\,1, q, r, s)$.

The Tutte polynomial of the graph $\overline{G}(q\,r\,s)$ (Fig. 32a) is

$$T(\overline{G}(q\,r\,s)) = \overline{G}(r)\overline{G}(s) + (y\frac{y^{q-1}-1}{y-1} + 1)\overline{G}(r+s),$$

and the Tutte polynomial of the graph $G(p\,\overline{q}\,\overline{r}\,\overline{s})$ (Fig. 32b) is

$$T(G(p\,\overline{q}\,\overline{r}\,\overline{s})) = \frac{x^p-1}{x-1}(\frac{y^q-1}{y-1}+x-1)(\frac{y^r-1}{y-1}+x-1)(\frac{y^s-1}{y-1}+x-1)+T(\overline{G}(q\,r\,s)).$$

Since graph $G(p\,1,q,r,s)$ (Fig. 32c) resolves into the graph $G(p\,\overline{q}\,\overline{r}\,\overline{s})$ and the block sum of graphs $\overline{G}(q\,r\,s)$ and $G(p)$, the general formula for the Tutte polynomial of the graphs $G(p\,1,q,r,s)$ is

$$T(G(p\,1,q,r,s)) = T(G(p\,\overline{q}\,\overline{r}\,\overline{s})) + T(\overline{G}(q\,r\,s))T(G(p)).$$

2.39. *Family p q,r 1,s*

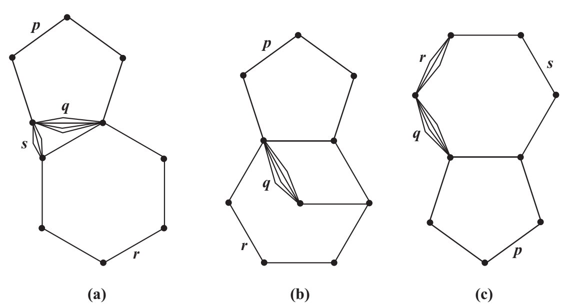


Figure 33: Graphs (a) $G(p\,q,r\,1,s)$; (b) $G(p\,1\,1,q\,1,r)$; (c) $G(p\,1,q,r+s)$.

Graph $G(p\,q,r\,1,s)$ (Fig. 33a) resolves into the graph $G(p\,q\,r\,s)$ and the block sum of graphs $G(p\,(q+s))$ and $G(r)$. The general formula for the Tutte polynomial of the graphs $G(p\,q,r\,1,s)$ is

$$T(G(p\,q,r\,1,s)) = T(G(p\,q\,r\,s)) + T(G(p\,(q+s)))T(G(r)).$$

2.40. *Family p 1 1,q 1,r*

Graph $G(p\,1\,1,q\,1,r)$ (Fig. 33b) resolves into the graph $G(r\,q\,1\,1\,p)$ and the block sum of graphs $G(p\,1\,(r+1))$ and $G(\overline{q})$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,1,q\,1,r)$ is

$$T(G(p\,1\,1,q\,1,r)) = T(G(r\,q\,1\,1\,p)) + T(G(p\,1\,(r+1)))T(G(\overline{q})).$$

2.41. *Family p 1,q,r+s*

Graph $G(p\,1, q, r+s)$ (Fig. 33c) resolves into the graph $G((p+s)\,\overline{q}\,\overline{r})$ and the block sum of graphs $G(s\,\overline{q}\,\overline{r})$ and $G(p)$. The general formula for the Tutte polynomial of the graphs $G(p\,1, q, r+s)$ is

$$T(G(p\,1, q, r+s)) = T(G((p+s)\,\overline{q}\,\overline{r})) + T(G(s\,\overline{q}\,\overline{r}))T(G(p)).$$

2.42. *Family p,q,r,s+*

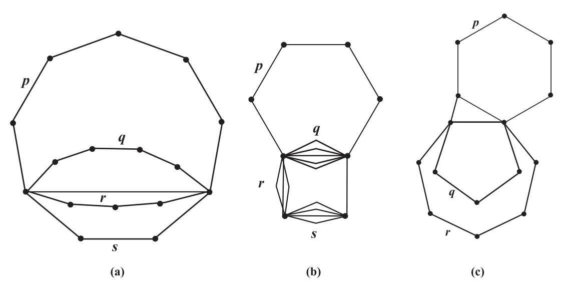


Figure 34: Graphs (a) $G(p, q, r, s+)$; (b) $G(p\,q, r, s+)$; (c) $G(p\,1\,1, q, r+)$.

Graph $G(p, q, r, s+)$ (Fig. 34a) resolves into the graph $G(p, q, r, s)$ and the block sum of graphs $G(p)$, $G(q)$, $G(r)$ and $G(s)$. The general formula for the Tutte polynomial of the graphs $G(p, q, r, s+)$ is

$$T(G(p, q, r, s+)) = T(G(p, q, r, s)) + T(G(p))T(G(q))T(G(r))T(G(s)).$$

2.43. *Family p q,r,s+*

Graph $G(p\,q, r, s+)$ (Fig. 34b) resolves into the graph $G(p\,q, r, s)$ and the block sum of graphs $G(p\,q)$, $G(\overline{r})$, and $G(\overline{s})$. The general formula for the Tutte polynomial of the graphs $G(p\,q, r, s+)$ is

$$T(G(p\,q, r, s+)) = T(G(p\,q, r, s)) + T(G(p\,q))T(G(\overline{r}))T(G(\overline{s})).$$

2.44. *Family p 1 1,q,r+*

Graph $G(p\,1\,1, q, r+)$ (Fig. 34c) resolves into the graph $G(p, q, r{+}{+}) = G(p, q, r+2)$ and the block sum of graphs $G(q\,1\,r)$ and $G((p+1))$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,1, q, r+)$ is

$$T(G(p\,1\,1, q, r+)) = T(G(p, q, r+2)) + T(G(q\,1\,r))T(G((p+1))).$$

2.45. *Family (p,q+) (r,s)*

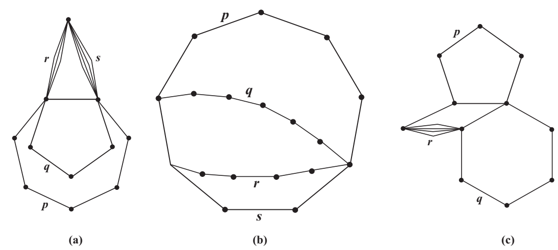


Figure 35: Graphs (a) $G((p,q+)\,(r,s))$; (b) $G((p,q)\,1\,(r,s))$; (c) $G(p\,1,q\,1,r+)$.

Graph $G((p,q+)\,(r,s))$ (Fig. 35a) resolves into the graph $G((p,q)\,(r,s))$ and the block sum of graphs $G(p)$, $G(q)$ and $G((\overline{r+s}))$. The general formula for the Tutte polynomial of the graphs $G((p,q+)\,(r,s))$ is

$$T(G((p,q+)\,(r,s))) = T(G((p,q)\,(r,s))) + T(G(p))T(G(q))T(G((\overline{r+s}))).$$

2.46. *Family (p,q) 1 (r,s)*

Graph $G((p,q)\,1\,(r,s))$ (Fig. 35b) resolves into the graph $G(p,q,r,s)$ and the block sum of graphs $G((p+q))$ and $G((r+s))$. The general formula for the Tutte polynomial of the graphs $G((p,q)\,1\,(r,s))$ is

$$T(G((p,q)\,1\,(r,s))) = T(G(p,q,r,s)) + T(G((p+q)))T(G((r+s))).$$

2.47. *Family p 1,q 1,r+*

Graph $G(p\,1,q\,1,r+)$ (Fig. 35c) resolves into the graph $G(p\,1,q\,1,r)$ and the block sum of graphs $G((p+1))$, $G((q+1))$, and $G(\overline{r})$. The general formula for the Tutte polynomial of the graphs $G(p\,1,q\,1,r+)$ is

$$T(G(p\,1,q\,1,r+)) = T(G(p\,1,q\,1,r)) + T(G((p+1)))T(G((q+1))T(G(\overline{r})).$$

2.48. *Family (p 1,q) (r,s)*

Graph $G((p\,1,q)\,(r,s))$ (Fig. 36a) resolves into the graph $G'(p\,\overline{q}\,r\,s)$ (Fig. 36b) and the block sum of graphs $G(s\,q\,r)$, and $G(p)$. The Tutte polynomial of the graphs $G'(p\,\overline{q}\,r\,s)$ is

$$T(G'(p\,\overline{q}\,r\,s)) = \frac{y^q-1}{y-1}T(G(p,r,s) + x^p(\frac{x^{r+s}-1}{x-1}+y-1),$$

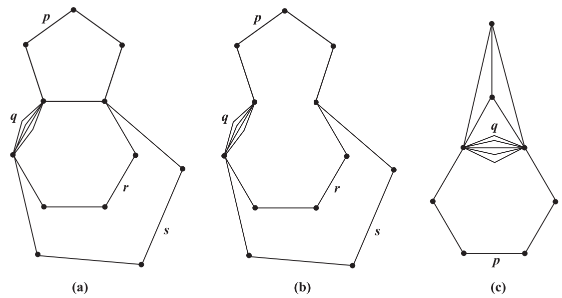


Figure 36: Graphs (a) $G((p\,1,q)\,(r,s))$; (b) $G'(p\,\overline{q}\,r\,s)$; (c) $G(.p\,q)$.

and the general formula for the Tutte polynomial of the graphs $G((p\,1,q)\,(r,s))$ is

$$T(G((p\,1,q)\,(r,s))) = T(G'(p\,\overline{q}\,r\,s)) + T(G(s\,q\,r))T(G(p)).$$

2.49. *Family .p q*

Graph $G(.p\,q)$ (Fig. 36c) resolves into the graphs $G(p\,q,2,2)$ and $G(p,2,2+q)$. The general formula for the Tutte polynomial of the graphs $G(.p\,q)$ is

$$T(G(.p\,q)) = T(G(p\,q,2,2)) + T(G(p,2,2+q)).$$

2.50. *Family .p 1 1*

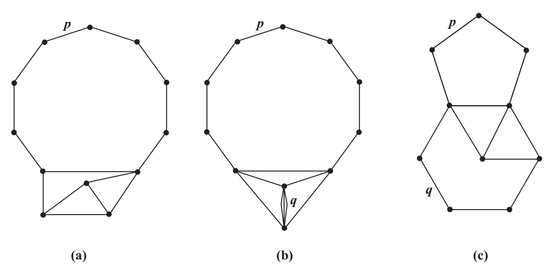


Figure 37: Graphs (a) $G(.p\,1\,1)$; (b) $G(.p\,1:q)$; (c) $G(.p\,1.q)$.

Graph $G(.p\,1\,1)$ (Fig. 37a) resolves into the graph $G(.p\,1)$ and the block sum of the graph $G((p+1))$ and the graph which consists from two triangles with the common edge. The general formula for the Tutte polynomial of the graphs $G(.p\,1\,1)$ is

$$T(G(.p\,1\,1)) = T(G(.p\,1)) + T(G((p+1)))(x + 2x^2 + x^3 + y + 2xy + y^2).$$

2.51. *Family .p 1:q*

Graph $G(.p\,1:q)$ (Fig. 37b) resolves into the graph $G(.p:q\,0)$ and the block sum of the graphs $G(p)$ and $\overline{G}(q\,2\,2)$. The general formula for the Tutte polynomial of the graphs $G(.p\,1:q)$ is

$$T(G(.p\,1:q)) = T(G(.p:q\,0)) + T(G(p))T(G(q\,2\,2)).$$

2.52. *Family .p 1.q*

Graph $G(.p\,1.q)$ (Fig. 37c) resolves into the graphs $G(p\,1, q\,1, 2)$ and $G(p,(q+1),2,1)$. The general formula for the Tutte polynomial of the graphs $G(.p\,1.q)$ is

$$T(G(.p\,1.q)) = T(G(p\,1, q\,1, 2)) + T(G(p,(q+1),2,1)).$$

2.53. *Family .p 1:q 0*

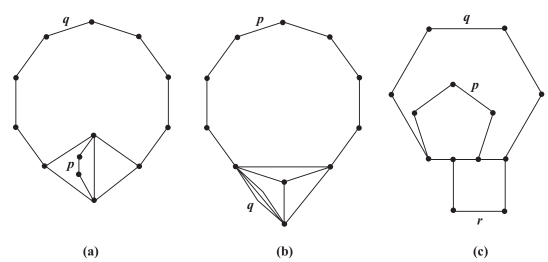


Figure 38: Graphs (a) $G(.p\,1:q\,0)$; (b) $G(.p\,1.q\,0)$; (c) $G(.p.q\,0.r)$.

Graph $G(.p\,1:q\,0)$ (Fig. 38a) resolves into the graph $G(.p:q)$ and the block sum of graphs $G(q,\overline{2},\overline{2})$ and $G(p)$. The Tutte polynomial of the graph $G(q,\overline{2},\overline{2})$ is

$$T(G(q,\overline{2},\overline{2})) = \frac{x^q-1}{x-1}(x+y)^2 + \frac{y^4-1}{y-1} + x - 1.$$

The general formula for the Tutte polynomial of the graphs $G(.p\,1:q\,0)$ is

$$T(G(.p\,1:q\,0)) = T(G(.p:q)) + T(G(q,\overline{2},\overline{2}))T(G(p)).$$

2.54. *Family .p 1.q 0*

Graph $G(.p\,1.q\,0)$ (Fig. 38b) resolves into the graph $G(.p.q)$ and the block sum of the graphs $G(1\overline{(q+1)}\,\overline{2})$ and $G(p)$. The Tutte polynomial of the graph $G(1\,\overline{(q+1)}\,\overline{2})$ is

$$T(G(1\,\overline{(q+1)}\,\overline{2})) = (\frac{y^{q+1}-1}{y-1} + x - 1)(x+y) + \frac{y^{q+3}-1}{y-1} + x - 1.$$

The general formula for the Tutte polynomial of the graphs $G(.p\,1.q\,0)$ is

$$T(G(.p\,1.q\,0)) = T(G(.p.q)) + T(G(1\,\overline{(q+1)}\,\overline{2}))T(G(p)).$$

2.55. *Family .p.q 0.r*

Graph $G(.p.q\,0.r)$ (Fig. 38c) resolves into the graph $G((p+1),q,(r+1))$ and the other graph which resolves into the graph $G((p+q),r,1)$ and the block sum of graphs $G(q,r,1)$ and $G(p)$. The general formula for the Tutte polynomial of the graphs $G(.p.q\,0.r)$ is

$$T(G(.p.q\,0.r)) = T(G((p+1),q,(r+1))) + T(G((p+q),r,1)) + T(G(q,r,1))T(G(p)).$$

2.56. *Family p:q 0:r 0*

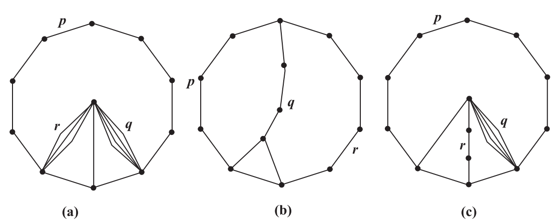


Figure 39: Graphs (a) $G(p:q\,0:r\,0)$; (b) $G(p\,0:q\,0:r\,0)$; (c) $G(.p.q.r)$.

Graph $G(p\,:\,q\,0\,:\,r\,0)$ (Fig. 39a) resolves into the graphs $G(p,\overline{q+1},\overline{r+1})$ and $G((p,2)\,(q,r))$. The general formula for the Tutte polynomial of the graphs $G(p:q\,0:r\,0)$ is

$$T(G(p:q\,0:r\,0)) = T(G(p,\overline{q+1},\overline{r+1})) + T(G((p,2)\,(q,r))).$$

2.57. *Family p 0:q 0:r 0*

Graph $G(p\,0:q\,0:r\,0)$ (Fig. 39b) resolves into the graph $G((p+1),q,(r+1))$ and the other graph which resolves into the graph $G(p,(q+1),r)$ and the block sum

of the graphs $G(p, q, r)$ and $G(\overline{1})$. The general formula for the Tutte polynomial forthe graphs $G(p\,0 : q\,0 : r\,0)$ is

$$T(G(p\,0 : q\,0 : r\,0)) = T(G((p+1), q, (r+1))) + T(G(p, (q+1), r)) + yT(G(p, q, r)).$$

2.58. *Family .p.q.r*

Graph $G(.p.q.r)$ (Fig. 39c) resolves into the graphs $G((p{+}1)\,1\,r\,q)$ and $G(p\,q\,1\,1\,r)$. The general formula for the Tutte polynomial of the graphs $G(.p.q.r)$ is

$$T(G(.p.q.r)) = T(G((p+1)\,1\,r\,q)) + T(G(p\,q\,1\,1\,r)).$$

2.59. *Family p:q:r*

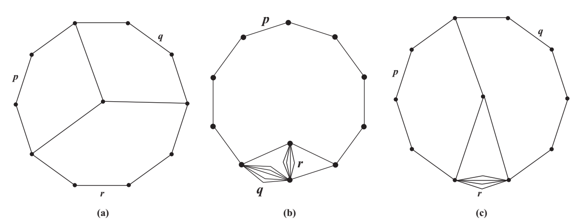


Figure 40: Graphs (a) $G(p : q : r)$; (b) $G(.p.q.r\,0)$; (c) $G(p : q : r\,0)$.

Graph $G(p : q : r)$ (Fig. 40a) resolves into the graphs $G((p + r), 2, q)$ and $G(p\,1\,q\,1\,r)$. The general formula for the Tutte polynomial of the graphs $G(p : q : r)$ is

$$T(G(p : q : r)) = T(G((p + r), 2, q)) + T(G(p\,1\,q\,1\,r)).$$

2.60. *Family .p.q.r 0*

Graph $G(.p.q.r\,0)$ (Fig. 40b) resolves into the graphs $G((p + 1)\,1, q, r)$ and $G(p\,q, (r + 1), 1)$. The general formula for the Tutte polynomial of the graphs $G(.p.q.r\,0)$ is

$$T(G(.p.q.r\,0)) = T(G((p + 1)\,1, q, r)) + T(G(p\,q, (r + 1), 1)).$$

2.61. *Family p:q:r 0*

Graph $G(p : q : r\,0)$ (Fig. 40c) resolves into the graphs $G((p + q)\,r\,2)$ and $G(p\,1, q\,1, r)$. The general formula for the Tutte polynomial of the graphs $G(p : q : r\,0)$ is

$$T(G(p:q:r\,0)) = T(G((p+q)\,r\,2)) + T(G(p\,1, q\,1, r)).$$

2.62. *Family .(p,q)*

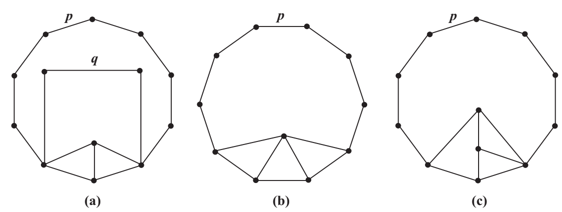


Figure 41: Graphs (a) $G(.(p,q))$; (b) $G(8^*p)$; (c) $G(8^*p\,0)$.

Graph $G(.(p,q))$ (Fig. 41a) resolves into the graphs $G(p,2,2,q)$ and $G((p,q)\,(2,2))$. The general formula for the Tutte polynomial of the graphs $G(.(p,q))$ is

$$T(G(.(p,q))) = T(G(p,2,2,q)) + T(G((p,q)\,(2,2))).$$

2.63. *Family 8** *p*

Graph $G(8^*p)$ (Fig. 41b) resolves into the graphs $G(2\,1\,p\,1\,2)$ and $G(.p:2\,0)$. The general formula for the Tutte polynomial of the graphs $G(8^*p)$ is

$$T(G(8^*p)) = T(G(2\,1\,p\,1\,2)) + T(G(.p:2\,0)).$$

2.64. *Family 8** *p 0*

Graph $G(8^*p\,0)$ (Fig. 41c) resolves into the graphs $G((p+1)\,1\,1\,1\,2)$ and $G(.p\,1:1)$. The general formula for the Tutte polynomial of the graphs $G(8^*p\,0)$ is

$$T(G(8^*p\,0)) = T(G((p+1)\,1\,1\,1\,2)) + T(G(.p\,1)).$$

2.65. *Family p q r s t*

The graphs $G(p\,q\,r\,s\,t)$ illustrated in Fig. 42a correspond to the link family ***p q r s t***. In order to obtain general formula for the Tutte polynomial we use the relations

$$T(G(p\,q\,r\,s\,t)) - T(G(p\,(q-1)\,r\,s\,t)) = y^{q-1}T(G(p))T(G(r\,s\,t)).$$

The general formula for the Tutte polynomial of the graphs $G(p\,q\,r\,s\,t)$ is

$$T(G(p\,q\,r\,s\,t)) = \frac{y^q-1}{y-1}T(G(p))T(G(r\,s\,t)) + T(G((p+r)\,s\,t)).$$

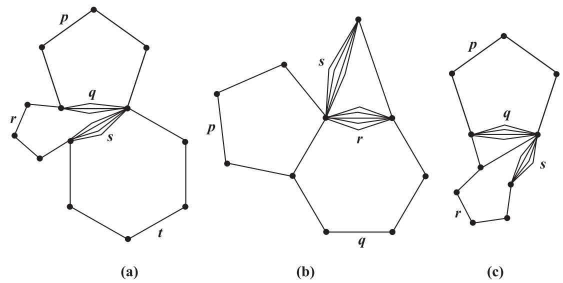


Figure 42: Graphs (a) $G(p\,q\,r\,s\,t)$; (b) $G(p\,1\,q\,r\,1\,s)$; (c) $G(p\,q\,1\,1\,r\,s)$.

2.66. *Family p 1 q r 1 s*

Graph $G(p\,1\,q\,r\,1\,s)$ (Fig. 42b) resolves into the graph $G((p+q)\,r\,1\,s)$ and the block sum of the graphs $G(q\,r\,1\,s)$ and $G(p)$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,q\,r\,1\,s)$ is

$$T(G(p\,1\,q\,r\,1\,s)) = T(G((p+q)\,r\,1\,s)) + T(G(q\,r\,1\,s))T(G(p)).$$

2.67. *Family p q 1 1 r s*

The graph $G(p\,q\,1\,1\,r\,s)$ illustrated in Fig. 42c corresponds to the link family ***p q 1 1 r s***. The graph $G(p\,q\,1\,1\,r\,s)$ resolves into the graph $G'(p\,(\overline{q+1})\,r\,\overline{s})$ and the block sum of the graphs $G(p\,q)$ and $G((r+1)\,s)$. In order to obtain general formula for the Tutte polynomial of the graph $G'(p\,\overline{q}\,r\,\overline{s})$ (Fig. 42d) we use the relations

$$T(G'(p\,\overline{q}\,r\,\overline{s})) - T(G'(p\,(\overline{q-1})\,r\,\overline{s})) = y^{q-1}T(G(p))T(G(r\,s)).$$

The Tutte polynomial of the graphs $G'(p\,\overline{q}\,r\,\overline{s})$ is

$$T(G'(p\,\overline{q}\,r\,\overline{s})) = \frac{y^q-1}{y-1}T(G(p))T(G(r\,s)) + T(G((p+r)\,s)),$$

and the general formula for the Tutte polynomial of the graphs $G(p\,q\,1\,1\,r\,s)$ is

$$T(G(p\,q\,1\,1\,r\,s)) = T(G'(p\,(\overline{q+1})\,r\,\overline{s})) + T(G(p\,q))T(G((r+1)\,s)).$$

2.68. *Family p q 1 r 1 s*

Graph $G(p\,q\,1\,r\,1\,s)$ (Fig. 43a) resolves into the graph $G(p\,q\,1\,(r+s))$ and the block sum of the graphs $G(p\,q\,1\,r)$ and $G(\overline{s})$. The general formula for the Tutte polynomial of the graphs $G(p\,q\,1\,r\,1\,s)$ is

$$T(G(p\,q\,1\,r\,1\,s)) = T(G(p\,q\,1\,(r+s))) + T(G(p\,q\,1\,r))T(G(\overline{s})).$$

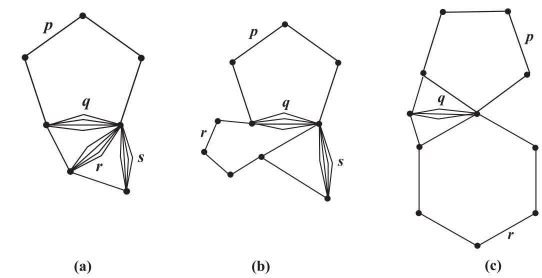


Figure 43: Graphs (a) $G(p\,q\,1\,r\,1\,s)$; (b) $G(p\,q\,r\,1\,1\,s)$; (c) $G(p\,1\,1\,q\,1\,1\,r)$.

2.69. *Family p q r 1 1 s*

Graph $G(p\,q\,r\,1\,1\,s)$ (Fig. 43b) resolves into the graph $G'(p\,\overline{q}\,(r+1)\,\overline{s})$ and the block sum of the graphs $G(p\,q\,r)$ and $G((\overline{s+1}))$. The general formula for the Tutte polynomial of the graphs $G(p\,q\,r\,1\,1\,s)$ is

$$T(G(p\,q\,r\,1\,1\,s)) = T(G'(p\,\overline{q}\,(r+1)\,\overline{s})) + T(G(p\,q\,r))T(G((\overline{s+1}))).$$

2.70. *Family p 1 1 q 1 1 r*

Graph $G(p\,1\,1\,q\,1\,1\,r)$ (Fig. 43c) resolves into the graph $G((p+1)\,q\,1\,1\,r)$ and the block sum of the graphs $G(r\,1\,1\,(q+1))$ and $G(p)$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,1\,q\,1\,1\,r)$ is

$$T(G(p\,1\,1\,q\,1\,1\,r)) = T(G((p+1)\,q\,1\,1\,r)) + T(G(r\,1\,1\,(q+1)))T(G(p)).$$

2.71. *Family p 1 q 1 1 1 r*

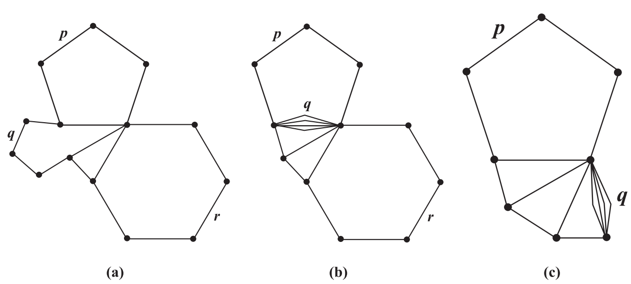


Figure 44: Graphs (a) $G(p\,1\,q\,1\,1\,1\,r)$; (b) $G(p\,q\,1\,1\,1\,1\,r)$; (c) $G(p\,1\,1\,1\,1\,1\,1\,r)$.

Graph $G(p\,1\,q\,1\,1\,1\,r)$ (Fig. 44a) resolves into the graph $G((p+q)\,1\,1\,1\,r)$ and the block sum of the graphs $G(q\,1\,1\,1\,r)$ and $G(p)$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,q\,1\,1\,1\,r)$ is

$$T(G(p\,1\,q\,1\,1\,1\,r)) = T(G((p+q)\,1\,1\,1\,r)) + T(G(q\,1\,1\,1\,r))T(G(p)).$$

2.72. *Family p q 1 1 1 1 r*

Graph $G(p\,q\,1\,1\,1\,1\,r)$ (Fig. 44b) resolves into the graph $G(p\,q\,1\,1\,(r+1))$ and the block sum of the graphs $G(p\,q\,1\,1\,1)$ and $G(r)$. The general formula for the Tutte polynomial of the graphs $G(p\,q\,1\,1\,1\,1\,r)$ is

$$T(G(p\,q\,1\,1\,1\,1\,r)) = T(G(p\,q\,1\,1\,(r+1))) + T(G(p\,q\,1\,1\,1))T(G(r)).$$

2.73. *Family p 1 1 1 1 1 1 q*

Graph $G(p\,1\,1\,1\,1\,1\,1\,q)$ (Fig. 44c) resolves into the graph $G((p+1)\,1\,1\,1\,1\,q)$ and the block sum of the graphs $\overline{G}(2\,1\,1\,1\,q)$ and $G(p)$, where $\overline{G}(2\,1\,1\,1\,q)$ is the dual of the graph $G(2\,1\,1\,1\,q)$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,1\,1\,1\,1\,1\,q)$ is

$$T(G(p\,1\,1\,1\,1\,1\,1\,q)) = T(G((p+1)\,1\,1\,1\,1\,q)) + T(\overline{G}(2\,1\,1\,1\,q))T(G(p)).$$

2.74. *Family p,q,r,s,t*

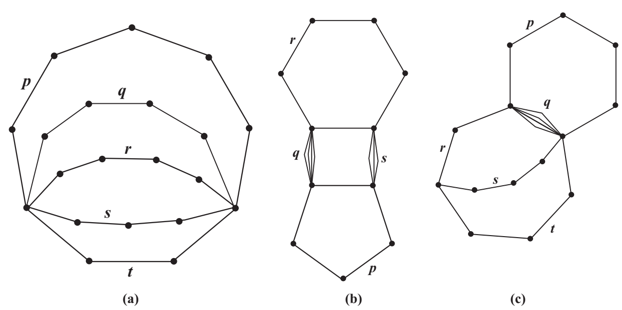


Figure 45: Graphs (a) $G(p,q,r,s,t)$; (b) $G(p\,1,q,r\,1,s)$; (c) $G(p\,q\,r,s,t)$.

In order to obtain formula for the Tutte polynomial of the graph $G(p,q,r,s,t)$ (Fig. 45a) we use the relations

$$T(G(p,q,r,s,t)) - T(G(p,q,r,s,(t-1))) = x^{t-1}T(G(p,q,r,s)).$$

Since the Tutte polynomial of the graph $G(p,q,r,s,0)$ is

$$T(G(p,q,r,s,0)) = T(G(p))T(G(q))T(G(r))T(G(s)),$$

the general formula for the Tutte polynomial of the graphs $G(p,q,r,s,t)$ is

$$T(G(p,q,r,s,t)) = \frac{x^t-1}{x-1}T(G(p,q,r,s)) + T(G(p))T(G(q))T(G(r))T(G(s)).$$

2.75. *Family p 1,q,r 1,s*

In order to obtain formula for the Tutte polynomial of the graph $G(p\,1,q,r\,1,s)$ (Fig. 45b) we use the relations

$$T(G(p\,1,q,r\,1,s)) - T(G(p\,1,q,r\,1,(s-1))) = y^{s-1}T(G(p\,1,r\,1,q)).$$

Since the Tutte polynomial of the graph $G(p\,1,q,r\,1,0)$ is $T(G(p\,1,q,r\,1,0)) = T(G(p+1))T(G(\overline{q}))T(G(r+1))$, the general formula for the Tutte polynomial of the graphs $G(p\,1,q,r\,1,s)$ is

$$T(p\,1,q,r\,1,s) = \frac{y^s-1}{y-1}T(G(p\,1,r\,1,q)) + T(G(p+1))T(G(\overline{q}))T(G(r+1)).$$

2.76. *Family p q r,s,t*

In order to obtain formula for the Tutte polynomial of the graph $G(p\,q\,r,s,t)$ (Fig. 45c) we use the relations

$$T(G(p\,q\,r,s,t)) - T(G(p\,q\,r,s,(t-1))) = x^{t-1}T(G(p\,q\,(r+s))).$$

Since the Tutte polynomial of the graph $G(p\,q\,r,s,0)$ is $T(G(p\,q\,r,s,0)) = T(G(p\,q\,r))T(G(s))$, the general formula for the Tutte polynomial of the graphs $G(p\,q\,r,s,t)$ is

$$T(G(p\,q\,r,s,t)) = \frac{x^t-1}{x-1}T(G(p\,q\,(r+s))) + T(G(p\,q\,r))T(G(s)).$$

2.77. *Family p 1 1 q,r,s*

Graph $G(p\,1\,1\,q,r,s)$ (Fig. 46a) resolves into the graph $G((p+1)\,q,r,s)$ and the block sum of the graphs $\overline{G}((q+1)\,r\,s)$ and $G(p)$, where $\overline{G}((q+1)\,r\,s)$ is the dual of the graph $G((q+1)\,r\,s)$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,1\,q,r,s)$ is

$$T(G(p\,1\,1\,q,r,s)) = T(G((p+1)\,q,r,s)) + T(\overline{G}((q+1)\,r\,s))T(G(p)).$$

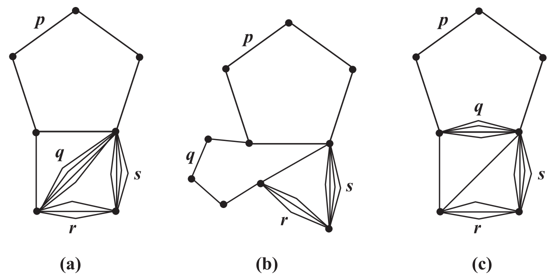


Figure 46: Graphs (a) $G(p\,1\,1\,q, r, s)$; (b) $G(p\,1\,q\,1, r, s)$; (c) $G(p\,q\,1\,1, r, s)$.

2.78. *Family p 1 q 1,r,s*

Graph $G(p\,1\,q\,1, r, s)$ (Fig. 46b) resolves into the graph $G((p+q)\,1, r, s)$ and the block sum of the graphs $G(q\,1, r, s)$ and $G(p)$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,q\,1, r, s)$ is

$$T(G(p\,1\,q\,1, r, s)) = T(G((p+q)\,1, r, s)) + T(G(q\,1, r, s))T(G(p)).$$

2.79. *Family p q 1 1,r,s*

Graph $G(p\,q\,1\,1, r, s)$ (Fig. 46c) resolves into the graph $G(p\,(q+1), s, r)$ and the block sum of the graphs $\overline{G}(s\,1\,r)$ and $G(p\,q)$. The general formula for the Tutte polynomial of the graphs $G(p\,q\,1\,1, r, s)$ is

$$T(G(p\,q\,1\,1, r, s)) = T(G(p\,(q+1), s, r)) + T(\overline{G}(s\,1\,r))T(G(p\,q)).$$

2.80. *Family p 1 1 1 1,q,r*

Graph $G(p\,1\,1\,1\,1, q, r)$ (Fig. 47a) resolves into the graph $G(p\,1\,1, q, r+)$ and the block sum of the graphs $G(p\,1\,2)$ and $G(q+r)$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,1\,1\,1, q, r)$ is

$$T(p\,1\,1\,1\,1, q, r) = T(G(p\,1\,1, q, r+)) + T(G(p\,1\,2))T(G(q+r)).$$

2.81. *Family p q,r s,t*

In order to obtain formula for the Tutte polynomial of the graph $G(p\,q, r\,s, t)$ (Fig. 47b) we use the relations

$$T(G(p\,q, r\,s, t)) - T(G(p\,q, r\,s, (t-1))) = y^{t-1}T(G(p\,(q+s)\,r)).$$

Since the Tutte polynomial of the graph $G(p\,q, r\,s, 0)$ is $T(G(p\,q, r\,s, 0)) = T(G(p\,q))T(G(r\,s))$, the general formula for the Tutte polynomial of the graphs $G(p\,q, r\,s, t)$ is

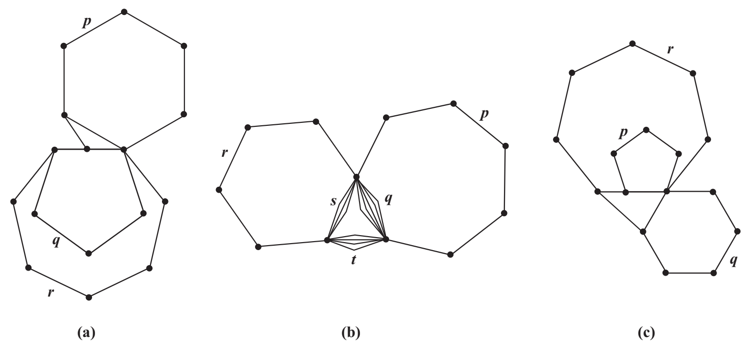


Figure 47: Graphs (a) $G(p\,1\,1\,1\,1,q,r)$; (b) $G(p\,q,r\,s,t)$; (c) $G(p\,1\,1,q\,1\,1,r)$.

$$T(G(p\,q,r\,s,t)) = \frac{y^t-1}{y-1}T(G(p\,(q+s)\,r)) + T(G(p\,q))T(G(r\,s)).$$

2.82. *Family p 1 1,q 1 1,r*

Graph $G(p\,1\,1,q\,1\,1,r)$ (Fig. 47c) resolves into the graph $G(q\,1\,1,p,r+)$ and the block sum of the graphs $G(q\,1\,(r+1))$ and $G(p+1)$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,1,q\,1\,1,r)$ is

$$T(G(p\,1\,1,q\,1\,1,r)) = T(G(q\,1\,1,p,r+)) + T(G(q\,1\,(r+1)))T(G(p+1)).$$

2.83. *Family p q,r 1,s 1*

Graph $G(p\,q,r\,1,s\,1)$ (Fig. 48a) resolves into the graph $G(p\,q\,r\,1\,s)$ and the block sum of the graphs $G(p\,(q+1)\,s)$ and $G(r)$. The general formula for the Tutte polynomial of the graphs $G(p\,q,r\,1,s\,1)$ is

$$T(G(p\,q,r\,1,s\,1)) = T(G(p\,q\,r\,1\,s)) + T(G(p\,(q+1)\,s))T(G(r)).$$

2.84. *Family p 1 1,q 1,r 1*

Graph $G(p\,1\,1,q\,1,r\,1)$ (Fig. 48b) resolves into the graph $G(p\,1\,1\,(q+r))$, the block sum of graphs $G(p\,1\,1\,q)$ and $G(\overline{r})$, and the block sum of graphs $G(p\,1\,2\,r)$ and $G(\overline{q})$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,1,q\,1,r\,1)$ is

$$T(G(p\,1\,1,q\,1,r\,1)) = T(G(p\,1\,1\,(q+r)) + T(G(p\,1\,1\,q))T(G(\overline{r})) + T(G(p\,1\,2\,r))T(G(\overline{q})).$$

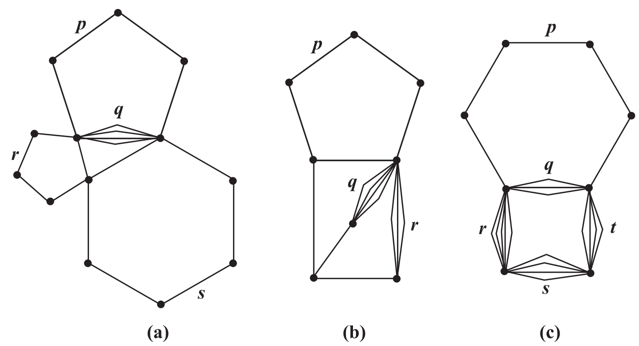


Figure 48: Graphs (a) $G(p\,q, r\,1, s\,1)$; (b) $G(p\,1\,1, q\,1, r\,1)$; (c) $G(p\,q, r, s, t)$.

2.85. *Family p q,r,s,t*

In order to obtain formula for the Tutte polynomial of the graph $G(p\,q, r, s, t)$ (Fig. 48c) we use the relations

$$T(G(p\,q, r, s, t)) - T(G(p\,q, r, s, (t-1))) = y^{t-1}T(G(p\,q, r, s)).$$

Since the Tutte polynomial of the graph $G(p\,q, r, s, 0)$ is

$$T(G(p\,q, r, s, 0)) = T(G(\overline{r}))T(G(\overline{s}))T(G(p\,q)),$$

the general formula for the Tutte polynomial of the graphs $G(p\,q, r, s, t)$ is

$$T(G(p\,q, r, s, t)) = \frac{y^t - 1}{y - 1}T(G(p\,q, r, s)) + T(G(\overline{r}))T(G(\overline{s}))T(G(p\,q)).$$

2.86. *Family p 1 1,q,r,s*

Graph $G(p\,1\,1, q, r, s)$ (Fig. 49a) resolves into the graph $G((p+1), q, r, s)$, the block sum of graphs $G(q, r, s)$ and $G(p)$, and the block sum of graphs $G(p)$, $G(q)$, $G(r)$, and $G(s)$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,1, q, r, s)$ is

$$T(G(p\,1\,1, q, r, s)) = T(G((p+1), q, r, s)) + T(G(q, r, s))T(G(p)) +$$

$$T(G(p))T(G(q))T(G(r))T(G(s)).$$

2.87. *Family p 1,q 1,r,s*

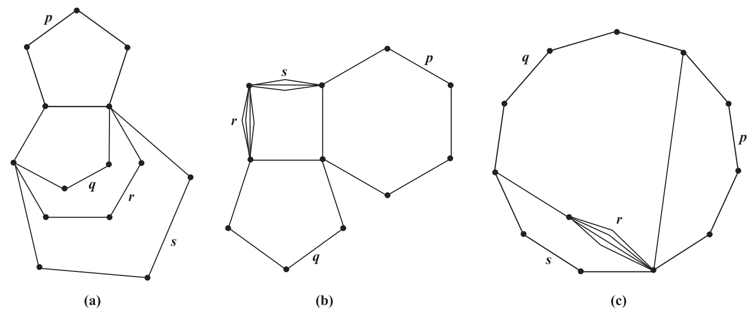


Figure 49: Graphs (a) $G(p\,1\,1,q,r,s)$; (b) $G(p\,1,q\,1,r,s)$; (c) $G(p\,1\,q,r\,1,s)$.

Graph $G(p\,1,q\,1,r,s)$ (Fig. 49b) resolves into the graph $G((p+q)\,\overline{r}\,\overline{s})$, the block sum of graphs $G(p\,1,r,s)$ and $G(q)$, and the block sum of graphs $G(q\,\overline{r}\,\overline{s})$ and $G(p)$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,1,q,r,s)$ is

$$T(G(p\,1,q\,1,r,s)) = T(G((p+q)\,\overline{r}\,\overline{s}))+T(G(p\,1,r,s))T(G(q))+T(G(q\,\overline{r}\,\overline{s}))T(G(p)).$$

2.88. *Family p 1 q,r 1,s*

Graph $G(p\,1\,q,r\,1,s)$ (Fig. 49c) resolves into the graph $G(s\,r\,q\,1\,p)$ and the block sum of the graphs $G(s+q,1,p)$ and $\overline{G}(r)$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,q,r\,1,s)$ is

$$T(G(p\,1\,q,r\,1,s)) = T(G(s\,r\,q\,1\,p)) + T(G(s+q,1,p))T(\overline{G}(r)).$$

2.89. *Family p q 1,r 1,s*

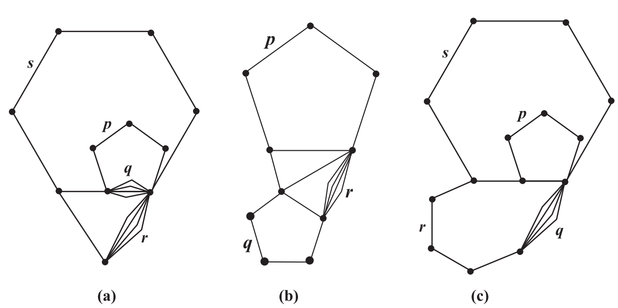


Figure 50: Graphs (a) $G(p\,q\,1,r\,1,s)$; (b) $G(p\,1\,1\,1,q\,1,r)$; (c) $G(p\,1\,1,q\,r,s)$.

Graph $G(p\,q\,1,r\,1,s)$ (Fig. 50a) resolves into the graph $G(p\,(q+r)\,s)$, the block sum of graphs $G(p\,q)$ and $G((s+1)\,r)$, and the block sum of graphs $G(p\,q\,s)$ and $G(\overline{r})$. The general formula for the Tutte polynomial of the graphs $G(p\,q\,1,r\,1,s)$ is

$$T(G(p\,q\,1,r\,1,s)) = T(G(p\,(q{+}r)\,s)){+}T(G(p\,q))T(G((s{+}1)\,r)){+}T(G(p\,q\,s))T(G(\overline{r})).$$

2.90. *Family p 1 1 1,q 1,r*

Graph $G(p\,1\,1\,1,q\,1,r)$ (Fig. 50b) resolves into the graph $G((p+1)\,1,q\,1,r)$ and the block sum of graphs $G(q\,1,2,r)$ and $G(p)$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,1\,1,q\,1,r)$ is

$$T(G(p\,1\,1\,1,q\,1,r)) = T(G((p+1)\,1,q\,1,r) + T(G(q\,1,2,r))T(G(p)).$$

2.91. *Family p 1 1,q r,s*

Graph $G(p\,1\,1,q\,r,s)$ (Fig. 50c) resolves into graph $\overline{G}(q\,r,s,(p+1))$, dual of the graph $G(q\,r,s,(p{+}1))$, the block sum of graphs $G((r{+}s)\,q)$ and $G(p)$, and the block sum of graphs $G(r\,q)$, $G(p)$ and $G(s)$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,1,q\,r,s)$ is

$$T(G(p\,1\,1,q\,r,s)) = T(\overline{G}(q\,r,s,(p+1)) + T(G((r+s)\,q))T(G(p))+$$

$$T(G(r\,q))T(G(s))T(G(p)).$$

2.92. *Family p,q,r,s+t*

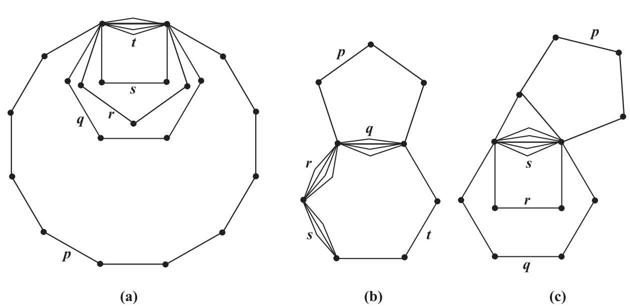


Figure 51: Graphs (a) $G(p,q,r,s+t)$; (b) $G(p\,q,r,s+t)$; (c) $G(p\,1\,1,q,r+s)$.

In order to obtain formula for the Tutte polynomial of the graph $G(p,q,r,s+t)$ (Fig. 51a) we use the relations

$$T(G(p,q,r,s+t))-T(G(p,q,r,s+(t-1))) = y^{t-1}T(G(p))T(G(q))T(G(r))T(G(s)).$$

Since the Tute polynomial of the graph $G(p,q,r,s,0)$ is $T(G(p,q,r,s,0)) = T(G(p,q,r,s))$, the general formula for the Tutte polynomial of the graphs $G(p,q,r,s+t)$ is

$$T(G(p,q,r,s+t)) = \frac{y^t-1}{y-1}T(G(p))T(G(q))T(G(r))T(G(s)) + T(G(p,q,r,s)).$$

2.93. *Family p q,r,s+t*

In order to obtain formula for the Tutte polynomial of the graph $G(p\,q,r,s+t)$ (Fig. 51b) we use the relations

$$T(G(p\,q,r,s+t)) - T(G(p\,q,r,s+(t-1))) = x^{t-1}T(G(p\,q))T(G(\overline{r}))T(G(\overline{s})).$$

Since the Tutte polynomial of the graph $G(p\,q,r,s+0)$ is $T(G(p\,q,r,s+0)) = T(G(p\,q,r,s))$, the general formula for the Tutte polynomial of the graphs $G(p\,q,r,s+t)$ is

$$T(G(p\,q,r,s+t)) = \frac{x^t-1}{x-1}T(G(p\,q))T(G(\overline{r}))T(G(\overline{s})) + T(G(p\,q,r,s)).$$

2.94. *Family p 1 1,q,r+s*

Graph $G(p\,1\,1,q,r+s)$ (Fig. 51c) resolves into the graph $G(\overline{s}\,q\,r\,(p+1))$, dual of the graph $G(s\,\overline{q}\,\overline{r}\,(\overline{p+1}))$ and the block sum of graphs $G(q\,(s+1)\,r)$ and $G(p)$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,1,q,r+s)$ is

$$T(G(p\,1\,1,q,r+s)) = T(G(\overline{s}\,q\,r\,(p+1))) + T(G(q\,(s+1)\,r))T(G(p)).$$

2.95. *Family p 1,q 1,r+s*

Graph $G(p\,1,q\,1,r+s)$ (Fig. 52a) resolves into the graph $G(q\,1,r,1+(p+s-1))$ and the block sum of the graphs $G(q\,1,r,1+(s-1))$ and $G(p)$. The general formula for the Tutte polynomial of the graphs $G(p\,1,q\,1,r+s)$ is

$$T(G(p\,1,q\,1,r+s)) = T(G(q\,1,r,1+(p+s-1))) + T(G(q\,1,r,1+(s-1)))T(G(p)).$$

2.96. *Family p 1,q 1,r 1+*

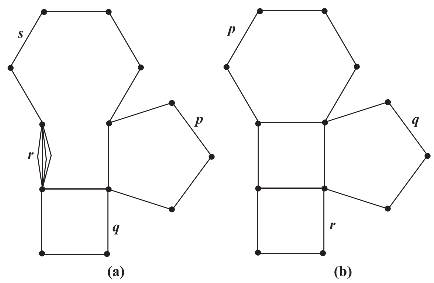


Figure 52: Graphs (a) $G(p\,1, q\,1, r+s)$; (b) $G(p\,1, q\,1, r\,1+)$; (c) $G(p\,1\,q, r, s+)$.

Graph $G(p\,1, q\,1, r\,1+)$ (Fig. 52b) resolves into the graph $G(p\,1\,(r+1)\,1\,q)$ and the block sum of the graphs $G(p\,1\,1\,1\,q)$ and $G(r)$. The general formula for the Tutte polynomial of the graphs $G(p\,1, q\,1, r\,1+)$ is

$$T(G(p\,1, q\,1, r\,1+)) = T(G(p\,1\,(r+1)\,1\,q)) + T(G(p\,1\,1\,1\,q))T(G(r)).$$

2.97. *Family p 1 q,r,s+*

Graph $G(p\,1\,q, r, s+)$ (Fig. 52c) resolves into the graph $G((p+q), 1, r, s)$ and the block sum of the graphs $G(q, 1, r, s)$ and $G(p)$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,q, r, s+)$ is

$$T(G(p\,1\,q, r, s+)) = T(G((p+q), 1, r, s)) + T(G(q, 1, r, s))T(G(p)).$$

2.98. *Family p q 1,r,s+*

Graph $G(p\,q\,1, r, s+)$ (Fig. 53a) resolves into the graph $G((\overline{q+1})\,r\,s\,p)$, dual of the graph $G((q+1)\,\overline{r}\,\overline{s}\,\overline{p})$ and the block sum of the graphs $G(p\,q)$ and $G(1, r, s)$. The general formula for the Tutte polynomial of the graphs $G(p\,q\,1, r, s+)$ is

$$T(G(p\,q\,1, r, s+)) = T(G((\overline{q+1})\,r\,s\,p)) + T(G(p\,q))T(G(1, r, s)).$$

2.99. *Family p 1 1 1,q,r+*

Graph $G(p\,1\,1\,1, q, r+)$ (Fig. 53b) resolves into the graph $G(p\,1\,1\,1, q, r)$ and the block sum of the graphs $G(p\,1\,2)$, $G(\overline{q})$, and $G(\overline{r})$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,1\,1, q, r+)$ is

$$T(G(p\,1\,1\,1, q, r+)) = T(G((p\,1\,1\,1, q, r)) + T(G(p\,1\,2))T(G(\overline{q}))T(G(\overline{r})).$$

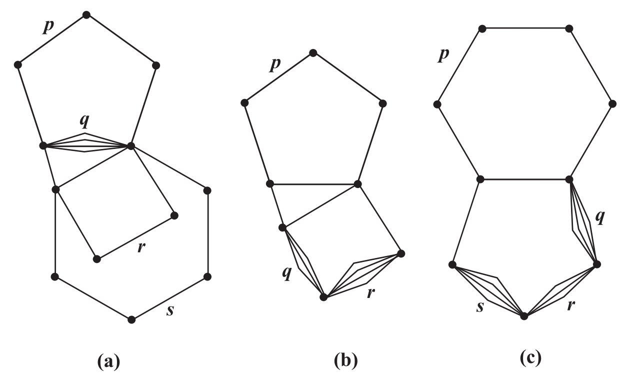


Figure 53: Graphs (a) $G(p\,q\,1, r, s{+})$; (b) $G(p\,1\,1\,1, q, r{+})$; (c) $G(p\,1, q, r, s{+})$.

2.100. *Family p 1,q,r,s+*

Graph $G(p\,1, q, r, s{+})$ (Fig. 53c) resolves into the graph $G(p\,1, q, r, s)$ and the block sum of the graphs $G(p+1)$, $G(\overline{q})$, $G(\overline{r})$, and $G(\overline{s})$. The general formula for the Tutte polynomial of the graphs $G(p\,1, q, r, s{+})$ is

$$T(G(p\,1, q, r, s{+})) = T(G((p\,1, q, r, s)) + T(G(p+1))T(G(\overline{q}))T(G(\overline{r}))T(G(\overline{s})).$$

2.101. *Family p q,r 1,s+*

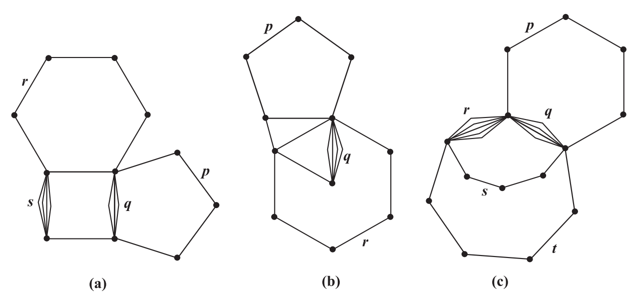


Figure 54: Graphs (a) $G(p\,q, r\,1, s{+})$; (b) $G(p\,1\,1, q\,1, r{+})$; (c) $G((p\,q, r)\,(s, t))$.

Graph $G(p\,q, r\,1, s{+})$ (Fig. 54a) resolves into the graph $G(p\,q, r\,1, s)$ and the block sum of the graphs $G(p\,q)$, $G(r+1)$, and $G(\overline{s})$. The general formula for the Tutte polynomial of the graphs $G(p\,q, r\,1, s{+})$ is

$$T(G(p\,q,r\,1,s+)) = T(G((p\,q,r\,1,s)) + T(G(p\,q))T(G(r+1))T(G(\overline{s})).$$

2.102. *Family p 1 1,q 1,r+*

Graph $G(p\,1\,1,q\,1,r+)$ (Fig. 54b) resolves into the graph $G(r\,(q+1)\,1\,1\,p)$ and the block sum of the graphs $G(p\,1\,1\,1\,r)$ and $G(\overline{q})$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,1,q\,1,r+)$ is

$$T(G(p\,1\,1,q\,1,r+)) = T(G((r\,(q+1)\,1\,1\,p)) + T(G(p\,1\,1\,1\,r))T(G(\overline{q})).$$

2.103. *Family (p q,r) (s,t)*

In order to obtain formula for the Tutte polynomial of the graph $G((p\,q,r)\,(s,t))$ (Fig. 54c) we use the relations

$$T(G((p\,q,r)\,(s,t))) - T(G((p\,q,r)\,(s,(t-1)))) = x^{t-1}T(G(p\,q\,s\,r)).$$

Since the Tutte polynomial of the graph $G((p\,q,r)\,(s,0))$ is $T(G((p\,q,r)\,(s,0))) = T(G(p\,(q+r)))T(G(s))$, the general formula for the Tutte polynomial of the graphs $G((p\,q,r)\,(s,t))$ is

$$T(G((p\,q,r)\,(s,t))) = \frac{x^t-1}{x-1}T(G(p\,q\,s\,r)) + T(G(p\,(q+r)))T(G(s)).$$

2.104. *Family (p 1 1,q) (r,s)*

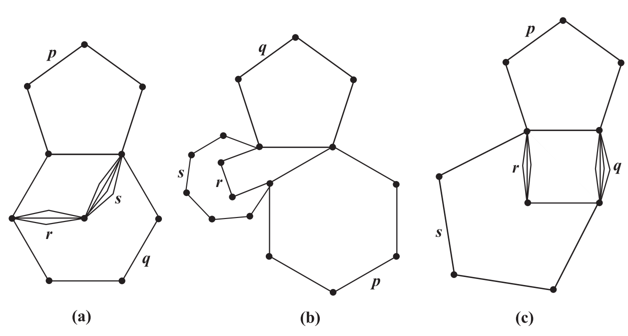


Figure 55: Graphs (a) $G((p\,1\,1,q)\,(r,s))$; (b) $G((p\,1,q\,1)\,(r,s))$; (c) $G((p\,1,q)\,(r\,1,s))$.

Graph $G((p\,1\,1,q)\,(r,s))$ (Fig. 55a) resolves into the graph $G((p+1,q)\,(r,s))$ and the block sum of the graphs $G((1,q)\,(r,s))$ and $G(p)$. The general formula for the Tutte polynomial of the graphs $G((p\,1\,1,q)\,(r,s))$ is

$$T(G((p\,1\,1,q)\,(r,s))) = T(G(((p+1,q)\,(r,s))) + T(G((1,q)\,(r,s)))T(G(p)).$$

2.105. *Family (p 1,q 1) (r,s)*

In order to obtain formula for the Tutte polynomial of the graph $G((p\,1,q\,1)\,(r,s))$ (Fig. 55b) we use the relations

$$T(G((p\,1,q\,1)\,(r,s))) - T(G((p\,1,q\,1)\,(r,(s-1)))) = x^{s-1}T(G(p\,1\,r\,1\,q)).$$

Since the Tutte polynomial of the graph $G((p\,1,q\,1)\,(r,0))$ is $T(G((p\,1,q\,1)\,(r,0))) = T(G(p\,2\,q))T(G(r))$, the general formula for the Tutte polynomial of the graphs $G((p\,1,q\,1)\,(r,s))$ is

$$T(G((p\,1,q\,1)\,(r,s)) = \frac{x^s-1}{x-1}T(G(p\,1\,r\,1\,q)) + T(G(p\,2\,q))T(G(r)).$$

2.106. *Family (p 1,q) (r 1,s)*

Graph $G((p\,1,q)\,(r\,1,s))$ (Fig. 55c) resolves into the graph $G(s\,r,p\,1,q)$ and the block sum of the graphs $G(p\,1\,s\,q)$ and $G(\overline{r})$. The general formula for the Tutte polynomial of the graphs $G((p\,1,q)\,(r\,1,s))$ is

$$T(G((p\,1,q)\,(r\,1,s))) = T(G(s\,r,p\,1,q)) + T(G(p\,1\,s\,q)T(G(\overline{r})).$$

2.107. *Family (p,q,r) (s,t)*

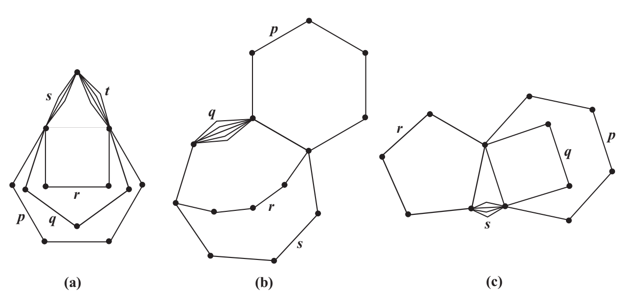


Figure 56: Graphs (a) $G((p,q,r)\,(s,t))$; (b) $G((p\,1,q+)\,(r,s))$; (c) $G((p,q+)\,(r\,1,s))$.

In order to obtain formula for the Tutte polynomial of the graph $G((p,q,r)\,(s,t))$ (Fig. 56a) we use the relations

$$T(G((p,q,r)\,(s,t)) - T(G(((p-1),q,r)\,(s,t))) = x^{p-1}T(G((q,r)\,(s,t))).$$

Since the Tutte polynomial of the graph $G((0,q,r)\,(s,t))$ is $T(G((0,q,r)\,(s,t))) = T(G(\overline{s+t}))T(G(r))T(G(q))$, the general formula for the Tutte polynomial of the graphs $G((p,q,r)\,(s,t))$ is

$$T(G((p,q,r)\,(s,t)) = \frac{x^p-1}{x-1}T(G((q,r)\,(s,t))) + T(G(\overline{s+t}))T(G(q))T(G(r)).$$

2.108. *Family (p 1,q+) (r,s)*

Graph $G((p\,1,q+)\,(r,s))$ (Fig. 56b) resolves into the graph $G((p\,1,q)\,(r,s))$ and the block sum of the graphs $G(r+s)$, $G(p+1)$, and $G(\overline{q})$. The general formula for the Tutte polynomial of the graphs $G((p\,1,q+)\,(r,s))$ is

$$T(G((p\,1,q+)\,(r,s))) = T(G((p\,1,q)\,(r,s))) + T(G(r+s))T(G(p+1))T(G(\overline{q})).$$

2.109. *Family (p,q+) (r 1,s)*

Graph $G((p,q+)\,(r\,1,s))$ (Fig. 56c) resolves into the graph $G((r\,1,s)\,(p,q))$ and the block sum of the graphs $G(r\,((s+1))$, $G(p)$, and $G(q)$. The general formula for the Tutte polynomial of the graphs $G((p,q+)\,(r\,1,s))$ is

$$T(G((p,q+)\,(r\,1,s))) = T(G((r\,1,s)\,(p,q))) + T(G(r\,((s+1)))T(G(p))T(G(q)).$$

2.110. *Family (p,q+) (r,s+)*

Graph $G((p,q+)\,(r,s+))$ (Fig. 57a) resolves into the graph $G((p,q+)\,(r,s))$ and the block sum of the graphs $G(p,q,1)$, $G(\overline{r})$, and $G(\overline{s})$. The general formula for the Tutte polynomial of the graphs $G((p,q+)\,(r,s+))$ is

$$T(G((p,q+)\,(r,s+)) = T(G((p,q+)\,(r,s)) + T(G(p,q,1)T(G(\overline{r}))T(G(\overline{s})).$$

2.111. *Family (p,q+r) (s,t)*

In order to obtain formula for the Tutte polynomial of the graph $G((p,q+r)\,(s,t))$ (Fig. 57b) we use the relations

$$T(G((p,q+r)\,(s,t)) - T(G((p,q+r)\,(s,(t-1)))) = y^{t-1}T(G(p\,(r+s)\,q)).$$

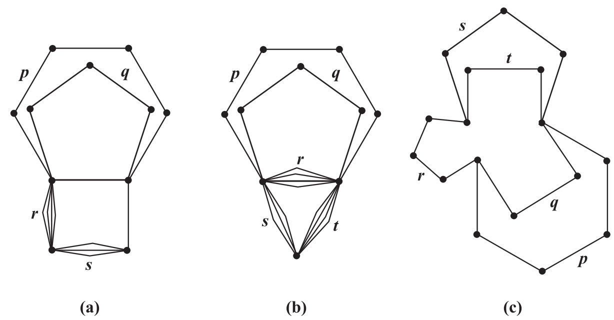


Figure 57: Graphs (a) $G((p,q+)\,(r,s+))$; (b) $G((p,q+r)\,(s,t))$; (c) $G((p,q)\,r\,(s,t))$.

Since the Tutte polynomial of the graph $G((p,q+r)\,(s,0))$ is $T(G((p,q+r)\,(s,0))) = T(G(p\,r\,q))T(G(\overline{s}))$, the general formula for the Tutte polynomial of the graphs $G((p,q+r)\,(s,t))$ is

$$T(G((p,q+r)\,(s,t)) = \frac{y^t-1}{y-1}T(G(p\,(r+s)\,q)) + T(G(p\,r\,q))T(G(\overline{s})).$$

2.112. *Family (p,q) r (s,t)*

In order to obtain formula for the Tutte polynomial of the graph $G((p,q)\,r\,(s,t))$ (Fig. 57c) we use the relations

$$T(G((p,q)\,r\,(s,t)) - T(G((p,q)\,r\,(s,(t-1)))) = x^{t-1}T(G(p,q,(r+s))).$$

Since the Tutte polynomial of the graph $G((p,q)\,r\,(s,0))$ is $T(G((p,q)\,r\,(s,0))) = T(G(p,q,r))T(G(s))$, the general formula for the Tutte polynomial of the graphs $G((p,q)\,r\,(s,t))$ is

$$T(G((p,q)\,r\,(s,t))) = \frac{x^t-1}{x-1}T(G(p,q,(r+s))) + T(G(p,q,r)))T(G(s)).$$

2.113. *Family (p 1,q) 1 (r,s)*

Graph $G((p\,1,q)\,1\,(r,s))$ (Fig. 58a) resolves into the graph $G(p\,1,q,r,s)$ and the block sum of the graphs $G(p\,(q+1))$ and $G(\overline{r+s})$. The general formula for the Tutte polynomial of the graphs $G((p\,1,q)\,1\,(r,s))$ is

$$T(G((p\,1,q)\,1\,(r,s)) = T(G(p\,1,q,r,s) + T(G(p\,(q+1)))T(G(\overline{r+s})).$$

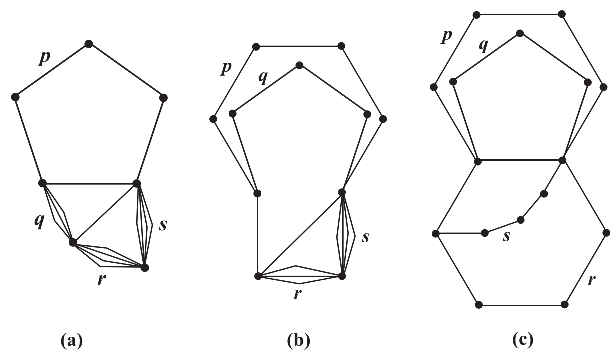


Figure 58: Graphs (a) $G((p\,1,q)\,1\,(r,s))$; (b) $G((p,q)\,1\,1\,(r,s))$; (c) $G((p,q+)\,1\,(r,s))$.

2.114. *Family (p,q) 1 1 (r,s)*

Graph $G((p,q)\,1\,1\,(r,s))$ (Fig. 58b) resolves into the graph $G((p,q+)\,(r,s))$ and the block sum of the graphs $\overline{G}(r\,1\,s)$ and $G(p+q)$. The general formula for the Tutte polynomial of the graphs is $G((p,q)\,1\,1\,(r,s))$ is

$$T(G((p,q)\,1\,1\,(r,s)) = T(G((p,q+)\,(r,s)) + T(\overline{G}(r\,1\,s))T(G(p+q)).$$

2.115. *Family (p,q+) 1 (r,s)*

Graph $G((p,q+)\,1\,(r,s))$ (Fig. 58c) resolves into the graph $G(p,q,r,s,1)$ and the block sum of the graphs $G(p,q,1)$ and $G(r+s)$. The general formula for the Tutte polynomial of the graphs is $G((p,q+)\,1\,(r,s))$ is

$$T(G((p,q+)\,1\,(r,s))) = T(G(p,q,r,s,1)) + T(G(p,q,1))T(G(r+s)).$$

2.116. *Family (p,q),r,(s,t)*

In order to obtain formula for the Tutte polynomial of the graph $G((p,q),r,(s,t))$ (Fig. 59a) we use the relations

$$T(G((p,q),r,(s,t))) - T(G((p,q),(r-1),(s,t))) = y^{r-1}T(G(p,q,s,t)).$$

Since the Tutte polynomial of the graph $G((p,q),0,(s,t))$ is $T(G((p,q),0,(s,t))) = T(G(p+q))T(G(s+t))$, the general formula for the Tutte polynomial of the graphs is $G((p,q),r,(s,t))$ is

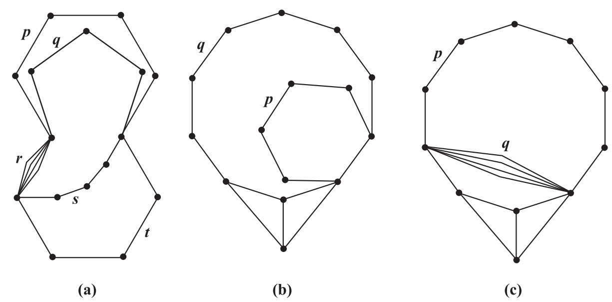


Figure 59: Graphs (a) $G((p,q),r,(s,t))$; (b) $G(.p\,1\,q)$; (c) $G(.p\,q\,1)$.

$$T(G((p,q),r,(s,t))) = \frac{y^r-1}{y-1}T(G(p,q,s,t)) + T(G(p+q))T(G(s+t)).$$

2.117. *Family .p 1 q*

Graph $G(.p\,1\,q)$ (Fig. 59b) resolves into the graph $G(.(p+q))$ and the block sum of the graphs $G(.q)$ and $G(p)$. The general formula for the Tutte polynomial of the graphs is $G(.p\,1\,q)$ is

$$T(G(.p\,1\,q)) = T(G(.(p+q))) + T(G(.q))T(G(p)).$$

2.118. *Family .p q 1*

In order to obtain formula for the Tutte polynomial of the graph $G(.p\,q\,1)$ (Fig. 59c) we use the relations

$$T(G(.p\,q\,1)) - T(G(.p\,(q-1)\,1)) = y^{q-1}T(G(p))T(G(.1)),$$

where $T(G(.1)) = 2x + 3x^2 + x^3 + 2y + 4xy + 3y^2 + y^3$.
Since the Tutte polynomial of the graph $G(.p\,0\,1)$ is $T(G(.p\,0\,1)) = T(G(.(p+1)))$, the general formula for the Tutte polynomial of the graphs is $G(.p\,q\,1)$ is

$$T(G(.p\,q\,1)) = \frac{y^q-1}{y-1}T(G(p))T(G(.1)) + T(G(.(p+1))).$$

2.119. *Family .p 1 1 1*

Graph $G(.p\,1\,1\,1)$ (Fig. 60a) resolves into the graph $G(.(p+1)\,1)$ and the block sum of the graphs $G(p)$ and G' (Fig. 60b) with the Tutte polynomial

$$T(G') = 2x + 3x^2 + x^3 + 2y + 5xy + x^2y + 4y^2 + 2xy^2 + 3y^3 + y^4.$$

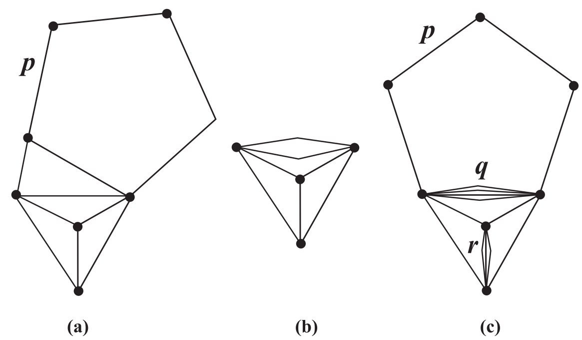


Figure 60: Graphs (a) $G(.p\,1\,1\,1)$; (b) G'; (c) $G(.p\,q:r)$.

The general formula for the Tutte polynomial of the graphs is $G(.p\,1\,1\,1)$ is

$$T(G(.p\,1\,1\,1)) = T(G(.(p+1)\,1)) + T(G')T(G(p)).$$

2.120. *Family .p q:r*

In order to obtain formula for the Tutte polynomial of the graph $G(.p\,q:r)$ (Fig. 60c) we use the relations

$$T(G(.p\,q:r)) - T(G(.p\,(q-1):r)) = y^{q-1}T(\overline{G}(1\,(r-1), 2, 2))T(G(p)).$$

Since the Tutte polynomial of the graph $G(.p\,(0):r)$ is $T(G(.p\,(0):r)) = T(G(.p:r\,0))$, the general formula for the Tutte polynomial of the graphs is $G(.p\,q:r)$ is

$$T(G(.p\,q:r)) = \frac{y^q-1}{y-1}T(G(p))T(\overline{G}(1\,(r-1), 2, 2)) + T(G(.p:r\,0)).$$

2.121. *Family .p q.r*

Graph $G(.p\,q.r)$ (Fig. 61a) resolves into the graphs $G(p\,q\,1\,1\,(r+1))$ and $G(p\,(q+2)\,r)$, and the block sum of the graphs $G(p\,q\,r)$ and $G(2)$ with the Tutte polynomial $T(G(2)) = x+y$.
The general formula for the Tutte polynomial of the graphs is $G(.p\,q.r)$ is

$$T(G(.p\,q.r)) = T(G(p\,q\,1\,1\,(r+1))) + T(G(p\,(q+2)\,r)) + T(G(p\,q\,r))T(G(2)).$$

2.122. *Family .p q:r 0*

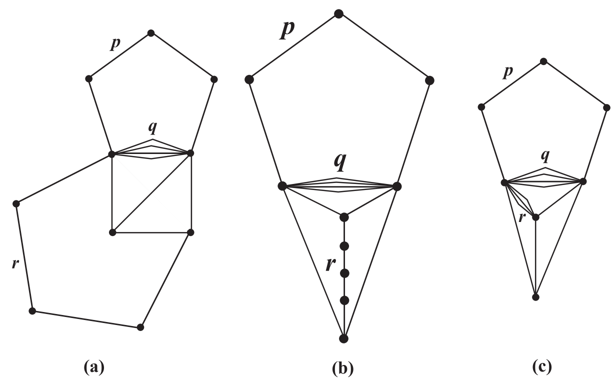


Figure 61: Graphs (a) $G(.p\,q.r)$; (b) $G(.p\,q:r\,0)$; (c) $G(.p\,q.r\,0)$.

In order to obtain formula for the Tutte polynomial of the graph $G(.p\,q:r\,0)$ (Fig. 61b) we use the relations

$$T(G(.p\,q:r\,0)) - T(G(.(p-1)\,q:r\,0)) = x^{p-1}T(G(.r:q\,0)).$$

Since the Tutte polynomial of the graph $G(.(0)\,q:r\,0)$ is $T(G(.(0)\,q:r\,0)) = y^qT(G(r\,\overline{2}\,\overline{2}))$, the general formula for the Tutte polynomial of the graphs $G(.p\,q:r\,0)$ is

$$T(G(.p\,q:r\,0)) = \frac{x^p-1}{x-1}T(G(.r:q\,0)) + y^qT(G(r\,\overline{2}\,\overline{2})).$$

2.123. *Family .p q.r 0*

Graph $G(.p\,q.r\,0)$ (Fig. 61c) resolves into the graphs $G(2\,r\,1\,q\,p)$ and $G(((p,1)+q)\,(r,2))$. The general formula for the Tutte polynomial of the graphs $G(.p\,q.r\,0)$ is

$$T(G(.p\,q.r\,0)) = T(G(2\,r\,1\,q\,p)) + T(G(((p,1)+q)\,(r,2))).$$

2.124. *Family .p 1 1:q*

Graph $G(.p\,1\,1:q)$ (Fig. 62a) resolves into the graph $G(.p\,1:q\,0)$ and the block sum of the graphs $G(2,q,2)$ and $G(p+1)$. The general formula for the Tutte polynomial of the graphs $G(.p\,1\,1:q)$ is

$$T(G(.p\,1\,1:q)) = T(G(.p\,1:q\,0)) + T(2,q,2)T(G(p+1)).$$

2.125. *Family .p 1 1.q*

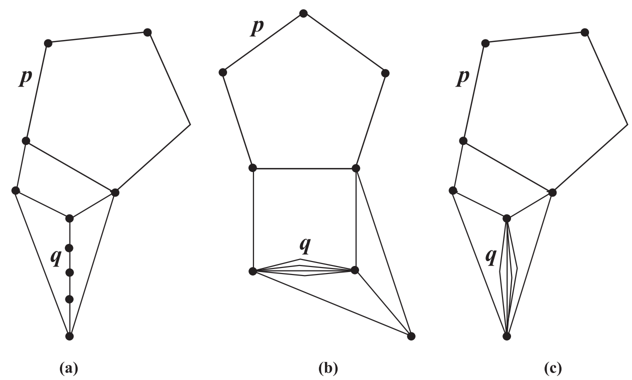


Figure 62: Graphs (a) $G(.p\,1\,1:q)$; (b) $G(.p\,1\,1.q)$; (c) $G(.p\,1\,1:q\,0)$.

Graph $G(.p\,1\,1.q)$ (Fig. 62b) resolves into the graph $G((p+1):1:q\,0)$ and the block sum of the graphs $G(.1:q\,0)$ and $G(p)$. The general formula for the Tutte polynomial of the graphs $G(.p\,1\,1.q)$ is

$$T(G(.p\,1\,1.q)) = T(G((p+1):1:q\,0)) + T(.1:q\,0)T(G(p)).$$

2.126. *Family .p 1 1:q 0*

Graph $G(.p\,1\,1:q\,0)$ (Fig. 62c) resolves into graph $G(.p\,1:q)$ and the block sum of graphs $G(2\,q\,2)$ and $G(p+1)$. The general formula for the Tutte polynomial of the graphs $G(.p\,1\,1:q\,0)$ is

$$T(G(.p\,1\,1:q\,0)) = T(G(.p\,1:q) + T(2\,q\,2)T(G(p+1)).$$

2.127. *Family .p 1 1.q 0*

Graph $G(.p\,1\,1.q\,0)$ (Fig. 63a) resolves into the graphs $G(p\,1,q\,1,2)$, $G(p,(q+1),2,1)$, and the block sum of the graphs $G((q+1)\,1\,2)$ and $G(p+1)$. The general formula for the Tutte polynomial of the graphs $G(.p\,1\,1.q\,0)$ is

$$T(G(.p\,1\,1.q\,0)) = T(G(p\,1,q\,1,2)) + T(G(p,(q+1),2,1)) + T((q+1)\,1\,2)T(G(p+1)).$$

2.128. *Family .p 1.q 0.r*

Graph $G(.p\,1.q\,0.r)$ (Fig. 63b) resolves into the graph $G(.p.q.r\,0)$ and the block sum of the graphs $\overline{G}((q+1),r,2)$ and $G(p)$. The general formula for the Tutte polynomial of the graphs $G(.p\,1.q\,0.r)$ is

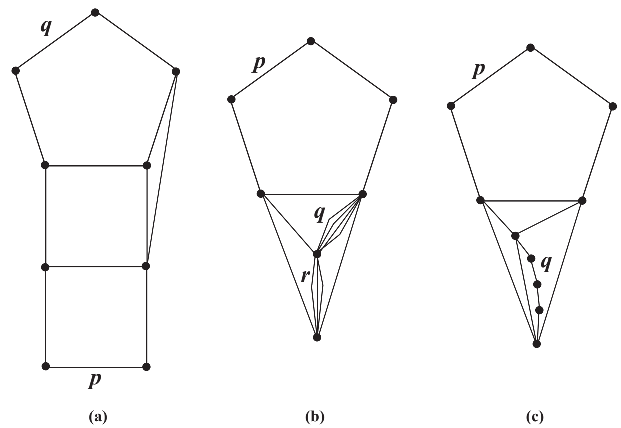


Figure 63: Graphs (a) $G(.p\,1\,1.q\,0)$; (b) $G(.p\,1.q\,0.r)$; (c) $G(.p\,1:q\,1)$.

$$T(G(.p\,1.q\,0.r)) = T(G(.p.q.r\,0)) + T(\overline{G}((q+1), r, 2))T(G(p)).$$

2.129. *Family .p 1:q 1*

Graph $G(.p\,1\,:\,q\,1)$ (Fig. 63c) resolves into the graph $G(.q\,1\,:\,p\,0)$ and the block sum of the graphs $G(q\,1, 2, 2)$ and $G(p)$. The general formula for the Tutte polynomial of the graphs $G(.p\,1:q\,1)$ is

$$T(G(.p\,1:q\,1)) = T(G(.q\,1:p\,0)) + T(q\,1, 2, 2)T(G(p)).$$

2.130. *Family .p 1.q 1*

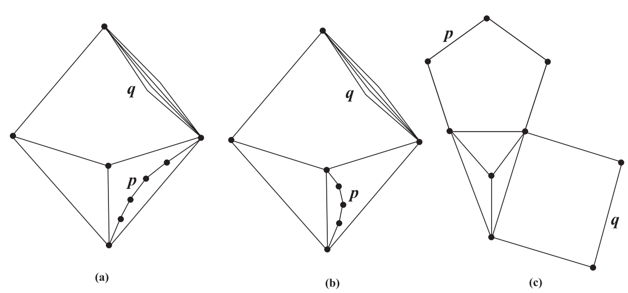


Figure 64: Graphs (a) $G(.p\,1.q\,1)$; (b) $G(.p\,1:q\,1\,0)$; (c) $G(.p\,1.q\,1\,0)$.

Graph $G(.p\,1.q\,1)$ (Fig. 64a) resolves into the graph $G(.p\,1.q\,0)$ and the block sum of the graphs $G(p\,1\,1\,1\,2)$ and $G(\overline{q})$. The general formula for the Tutte polynomial of the graphs $G(.p\,1.q\,1)$ is

$$T(G(.p\,1.q\,1)) = T(G(.p\,1.q\,0)) + T(p\,1\,1\,1\,2)T(G(\overline{q})).$$

2.131. *Family .p 1:q 1 0*

Graph $G(.p\,1 : q\,1\,0)$ (Fig. 64b) resolves into the graph $G(.p\,1 : q)$ and the block sum of the graphs $G(p, 2, 2, 1)$ and $G(\overline{q})$. The general formula for the Tutte polynomial of the graphs $G(.p\,1 : q\,1\,0)$ is

$$T(G(.p\,1 : q\,1\,0)) = T(G(.p\,1 : q)) + T(p, 2, 2, 1)T(G(\overline{q})).$$

2.132. *Family .p 1.q 1 0*

Graph $G(.p\,1.q\,1\,0)$ (Fig. 64c) resolves into the graph $G(.q\,1.p)$ and the block sum of the graphs $G(q\,2\,1\,2)$ and $G(p)$. The general formula for the Tutte polynomial of the graphs $G(.p\,1.q\,1\,0)$ is

$$T(G(.p\,1.q\,1\,0)) = T(G(.q\,1.p)) + T(q\,2\,1\,2)T(G(p)).$$

2.133. *Family .p 1 0.q.r*

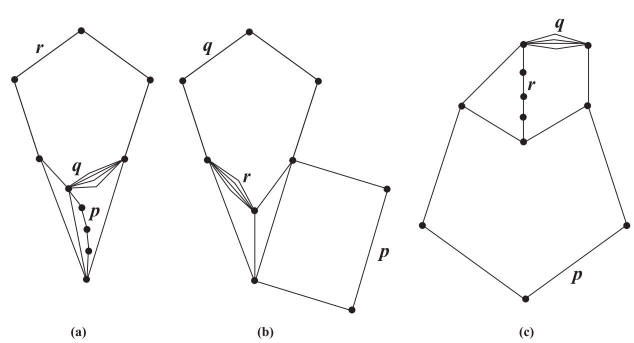


Figure 65: Graphs (a) $G(.p\,1\,0.q.r)$; (b) $G(p\,1.q.r)$; (c) $G(.p.q\,1\,0.r)$.

Graph $G(.p\,1\,0.q.r)$ (Fig. 65a) resolves into the graph $G(.r.q.p)$ and the block sum of the graphs $G(r\,(\overline{q+1})\,\overline{2})$ and $G(p)$. The general formula for the Tutte polynomial of the graphs $G(.p\,1\,0.q.r)$ is

$$T(G(.p\,1\,0.q.r)) = T(G(.r.q.p)) + T(G(r\,(\overline{q+1})\,\overline{2}))T(G(p)).$$

2.134. *Family .p 1.q.r*

Graph $G(.p\,1.q.r)$ (Fig. 65b) resolves into the graph $G(.p : r\,0.q\,0)$ and the block sum of the graphs $G(q\,1, r, 2)$ and $G(p)$. The general formula for the Tutte polynomial of the graphs $G(.p\,1.q.r)$ is

$$T(G(.p\,1.q.r)) = T(G(.p : r\,0.q\,0)) + T(G(q\,1, r, 2))T(G(p)).$$

2.135. *Family .p.q 1 0.r*

Graph $G(.p.q\,1\,0.r)$ (Fig. 65c) resolves into the graph $G(.p.q.r)$ and the block sum of the graphs $G((p+1)\,1\,(r+1))$ and $G(\overline{q})$. The general formula for the Tutte polynomial of the graphs $G(.p.q\,1\,0.r)$ is

$$T(G(.p.q\,1\,0.r)) = T(G(.p.q.r)) + T(G((p+1)\,1\,(r+1)))T(G(\overline{q})).$$

2.136. *Family .p.q 1.r*

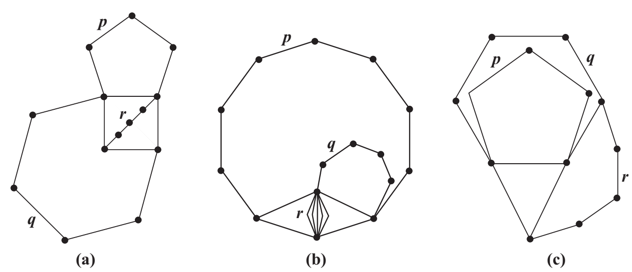


Figure 66: Graphs (a) $G(.p.q\,1.r)$; (b) $G(.p.q\,1.r\,0)$; (c) $G(.p\,1.q.r\,0)$.

Graph $G(.p.q\,1.r)$ (Fig. 66a) resolves into the graph $G(.p.q\,0.r)$ and the block sum of the graphs $G(p\,1\,1\,1\,r)$ and $G(q)$. The general formula for the Tutte polynomial of the graphs $G(.p.q\,1.r)$ is

$$T(G(.p.q\,1.r)) = T(G(.p.q\,0.r)) + T(G(p\,1\,1\,1\,r)T(G(q)).$$

2.137. *Family .p.q 1.r 0*

Graph $G(.p.q\,1.r\,0)$ (Fig. 66b) resolves into the graphs $G((p+1)\,1\,q\,r)$ and $G((p,q)\,(1,(r+1)))$, and the block sum of the graphs $G(p\,1\,1\,(r+1))$ and $G(q)$. The general formula for the Tutte polynomial of the graphs $G(.p.q\,1.r\,0)$ is

$$T(G(.p.q\,1.r\,0)) = T(G((p{+}1)\,1\,q\,r)){+}T(G((p,q)\,(1,(r{+}1)))){+}T(G(p\,1\,1\,(r{+}1)))T(G(q)).$$

2.138. *Family .p 1.q.r 0*

Graph $G(.p\,1.q.r\,0)$ (Fig. 66c) resolves into the graphs $G(p\,1\,1\,1\,(q+r))$ and $G(r\,1\,1,q,p+)$. The general formula for the Tutte polynomial of the graphs $G(.p\,1.q.r\,0)$ is

$$T(G(.p\,1.q.r\,0)) = T(G(p\,1\,1\,1\,(q+r))) + T(G(r\,1\,1,q,p+)).$$

2.139. *Family p 1 0:q 0:r 0*

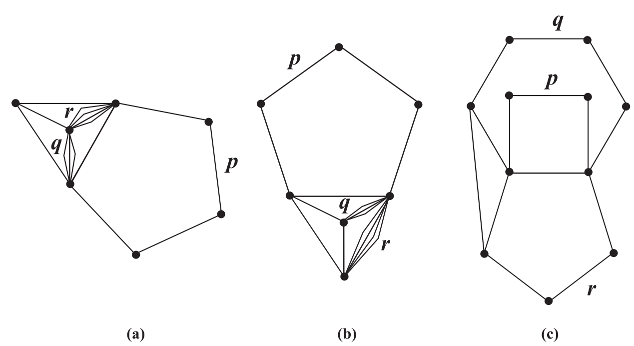


Figure 67: Graphs (a) $G(p\,1\,0:q\,0:r\,0)$; (b) $G(p\,1:q:r)$; (c) $G(p\,1:q\,0:r\,0)$.

Graph $G(p\,1\,0:q\,0:r\,0)$ (Fig. 67a) resolves into the graphs $G(p\,1,(q+1),(r+1))$ and $G((p,2+)\,(q,r))$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,0:q\,0:r\,0)$ is

$$T(G(p\,1\,0:q\,0:r\,0)) = T(G(p\,1,(q+1),(r+1))) + T(((p,2+)\,(q,r))).$$

2.140. *Family p 1:q:r*

Graph $G(p\,1:q:r)$ (Fig. 67b) resolves into the graphs $G(p\,q\,1\,r)$ and $G(p\,(\overline{q+r})\,\overline{2})$, the block sum of the graphs $G((p+1)\,r)$ and $G(\overline{q})$, and the block sum of the graphs $\overline{G}((q+1)\,1\,(r+1))$ and $G(p)$. The general formula for the Tutte polynomial of the graphs $G(p\,1:q:r)$ is

$$T(G(p\,1:q:r)) = T(G(p\,q\,1\,r)) + T(G(p\,(\overline{q+r})\,\overline{2})) + T(G((p+1)\,r))T(G(\overline{q}))+$$

$$T(\overline{G}((q+1)\,1\,(r+1)))T(G(p)).$$

2.141. *Family p 1:q 0:r 0*

Graph $G(p\,1 : q\,0 : r\,0)$ (Fig. 67c) resolves into the graphs $G(p\,0 : q\,0 : r\,0)$ and the block sum of the graphs $G(q\,1\,1\,1\,r)$ and $G(p)$. The general formula for the Tutte polynomial of the graphs $G(p\,1 : q\,0 : r\,0)$ is

$$T(G(p\,1 : q\,0 : r\,0)) = T(G(p\,0 : q\,0 : r\,0)) + T(G(q\,1\,1\,1\,r))T(G(p)).$$

2.142. *Family p:q:r 1 0*

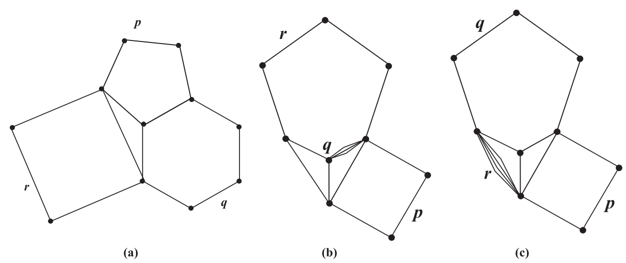


Figure 68: Graphs (a) $G(p : q : r\,1\,0)$; (b) $G(p\,1 : q : r\,0)$; (c) $G(p\,1\,0 : q : r\,0)$.

Graph $G(p : q : r\,1\,0)$ (Fig. 68a) resolves into the graphs $G(p\,1, q\,1, r\,1)$ and $G((p+q), r, 2, 1)$. The general formula for the Tutte polynomial of the graphs $G(p : q : r\,1\,0)$ is

$$T(G(p : q : r\,1\,0)) = T(G(p\,1, q\,1, r\,1)) + T(G((p+q), r, 2, 1)).$$

2.143. *Family p 1:q:r 0*

Graph $G(p\,1 : q : r\,0)$ (Fig. 68b) resolves into the graphs $G((r+1)\,q\,1\,1\,p)$ and $G((p, r+)\,(2, q))$. The general formula for the Tutte polynomial of the graphs $G(p\,1 : q : r\,0)$ is

$$T(G(p\,1 : q : r\,0)) = T(G((r+1)\,q\,1\,1\,p)) + T(G((p, r+)\,(2, q))).$$

2.144. *Family p 1 0:q:r 0*

Graph $G(p\,1\,0 : q : r\,0)$ (Fig. 68c) resolves into the graph $G(p : q : r\,0)$ and the block sum of the graphs $G(q\,r\,1\,2)$ and $G(p)$. The general formula for the Tutte polynomial of the graphs $G(p\,1\,0 : q : r\,0)$ is

$$T(G(p\,1\,0 : q : r\,0)) = T(G(p : q : r\,0)) + T(G(q\,r\,1\,2))T(G(p)).$$

2.145. *Family .p.q.r.s*

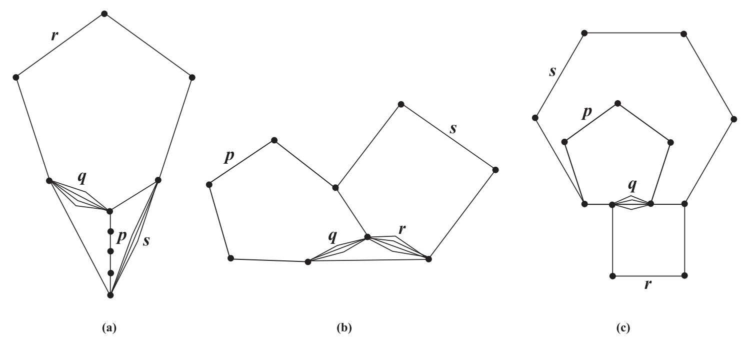


Figure 69: Graphs (a) $G(.p.q.r.s)$; (b) $G(.p.q.r\,0.s\,0)$; (c) $G(.p.q.r.s\,0)$.

In order to obtain formula for the Tutte polynomial of the graph $G(.p.q.r.s)$ (Fig. 69a) we use the relations

$$T(G(.p.q.r.s))-T(G(.p.q.(r-1).s)) = x^{r-1}(T(G(p\,q\,1\,s))+T(G((p+1)\,s))T(G(\overline{q}))).$$

Since the Tutte polynomial of the graph $G(.p.q.(0).s)$ is $T(G(.p.q.(0).s)) = T(G(p\,(\overline{q+1})\,(\overline{s+1})))$, the general formula for the Tutte polynomial of the graphs $G(.p.q.r.s)$ is

$$T(G(.p.q.r.s)) = \frac{x^r-1}{x-1}(T(G(p\,q\,1\,s))+T(G((p+1)\,s))T(G(\overline{q})))+T(G(p\,(\overline{q+1})\,(\overline{s+1}))).$$

2.146. *Family .p.q.r 0.s 0*

Graph $G(.p.q.r\,0.s\,0)$ (Fig. 69b) resolves into the graphs $G(p\,q\,1\,r\,s)$ and $G((p+s)\,1,q,r)$. The general formula for the Tutte polynomial of the graphs $G(.p.q.r\,0.s\,0)$ is

$$T(G(.p.q.r\,0.s\,0)) = T(G(p\,q\,1\,r\,s)) + T(G((p+s)\,1,q,r)).$$

2.147. *Family .p.q.r.s 0*

Graph $G(.p.q.r.s\,0)$ (Fig. 69c) resolves into the graphs $G((r+s)\,1\,p\,q)$ and $G(p,r,s+q)$, and the block sum of the graphs $G(r\,q)$ and $G(p+s)$. The general formula for the Tutte polynomial of the graphs $G(.p.q.r.s\,0)$ is

$$T(G(.p.q.r.s\,0)) = T(G((r+s)\,1\,p\,q)) + T(G(p,r,s+q)) + T(G(r\,q))T(G(p+s)).$$

2.148. *Family .p.q 0.r.s 0*

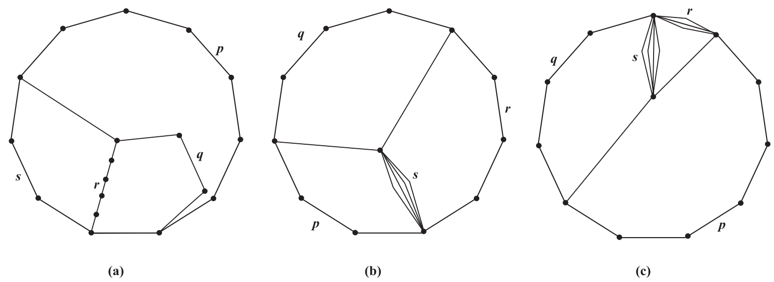


Figure 70: Graphs (a) $G(.p.q\,0.r.s\,0)$; (b) $G(p\,0.q.r\,0.s\,0)$; (c) $G(p\,0.q.r.s\,0)$.

Graph $G(.p.q\,0.r.s\,0)$ (Fig. 70a) resolves into the graphs $G(p,q,r,s)$ and $G((p+q)\,1\,(r+s))$, and the block sum of the graphs $G(p+s)$ and $G(q+r)$. The general formula for the Tutte polynomial of the graphs $G(.p.q\,0.r.s\,0)$ is

$$T(G(.p.q\,0.r.s\,0)) = T(G(p,q,r,s)) + T(G((p+q)\,1\,(r+s))) + T(G(p+s))T(G(q+r)).$$

2.149. *Family p 0.q.r 0.s 0*

Graph $G(p\,0.q.r\,0.s\,0)$ (Fig. 70b) resolves into the graphs $G(r\,s\,q\,1\,p)$ and $G((p+r)\,s\,q)$, and the block sum of the graphs $G(p+q+r)$ and $G(\overline{s})$. The general formula for the Tutte polynomial of the graphs $G(p\,0.q.r\,0.s\,0)$ is

$$T(G(p\,0.q.r\,0.s\,0)) = T(G(r\,s\,q\,1\,p)) + T(G((p+r)\,s\,q)) + T(G(p+q+r))T(G(\overline{s})).$$

2.150. *Family p 0.q.r.s 0*

Graph $G(p\,0.q.r.s\,0)$ (Fig. 70c) resolves into the graphs $G((p+q)\,r\,1\,s)$ and $G(q\,s,p\,1,r)$. The general formula for the Tutte polynomial of the graphs $G(p\,0.q.r.s\,0)$ is

$$T(G(p\,0.q.r.s\,0)) = T(G((p+q)\,r\,1\,s)) + T(G(q\,s,p\,1,r)).$$

2.151. *Family p.q 0.r.s 0*

Graph $G(p.q\,0.r.s\,0)$ (Fig. 71a) resolves into the graphs $G(p,q,(r+s))$ and $G(p,r,(q+s))$, the block sum of the graphs $G(p,q,r)$ and $G(s)$, and the block sum of the graph $G(q+r+s)$ and chain graph of the length p. The general formula for the Tutte polynomial of the graphs $G(p.q\,0.r.s\,0)$ is

$$T(G(p.q\,0.r.s\,0)) = T(G(p,q,(r+s))) + T(G(p,r,(q+s))) + T(G(p,q,r))T(G(s))+$$

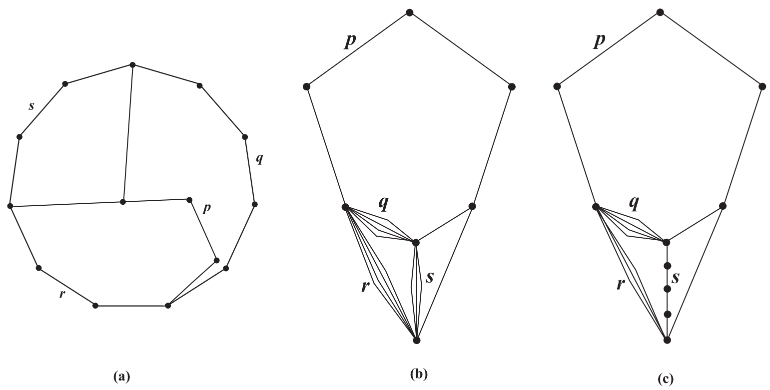


Figure 71: Graphs (a) $G(p.q\,0.r.s\,0)$; (b) $G(p.q.r\,0.s)$; (c) $G(p.q.r\,0.s\,0)$.

$$x^p T(G(q+r+s)).$$

2.152. *Family p.q.r 0.s*

Graph $G(p.q.r\,0.s)$ (Fig. 71b) resolves into the graphs $G((p+1)\,q,r,s)$ and $G(p\,r,q,(s+1))$. The general formula for the Tutte polynomial of the graphs $G(p.q.r\,0.s)$ is

$$T(G(p.q.r\,0.s)) = T(G((p+1)\,q,r,s)) + T(G(p\,r,q,(s+1))).$$

2.153. *Family p.q.r 0.s 0*

Graph $G(p.q.r\,0.s\,0)$ (Fig. 71c) resolves into the graphs $G((p+1)\,q\,s\,r)$ and $G(p\,r,s\,1,q)$. The general formula for the Tutte polynomial of the graphs $G(p.q.r\,0.s\,0)$ is

$$T(G(p.q.r\,0.s\,0)) = T(G((p+1)\,q\,s\,r)) + T(G(p\,r,s\,1,q)).$$

2.154. *Family p.q.r.s*

Graph $G(p.q.r.s)$ (Fig. 72a) resolves into the graphs $G((p+1)\,q\,r\,s)$ and $G((p,r)\,(q,(s+1)))$. The general formula for the Tutte polynomial of the graphs $G(p.q.r.s)$ is

$$T(G(p.q.r.s)) = T(G((p+1)\,q\,r\,s)) + T(G((p,r)\,(q,(s+1)))).$$

2.155. *Family p.q.r.s 0*

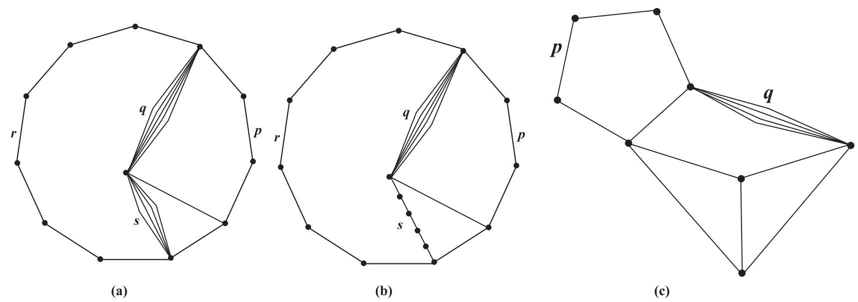


Figure 72: Graphs (a) $G(p.q.r.s)$; (b) $G(p.q.r.s\,0)$; (c) $G(.(p\,1,q))$.

Graph $G(p.q.r.s\,0)$ (Fig. 72b) resolves into the graphs $G'(s\,\overline{q}\,(p+1)\,r)$ and $G(p\,q\,r\,1\,s)$. The general formula for the Tutte polynomial of the graphs $G(p.q.r.s\,0)$ is

$$T(G(p.q.r.s\,0)) = T(G'(s\,\overline{q}\,(p+1)\,r)) + T(G(p\,q\,r\,1\,s)).$$

2.156. *Family .(p 1,q)*

Graph $G(.(p\,1,q))$ (Fig. 72c) resolves into the graphs $G((p\,1,q)\,(2,2))$ and $G(p\,1,q,2,2)$. The general formula for the Tutte polynomial of the graphs $G(.(p\,1,q))$ is

$$T(G(.(p\,1,q))) = T(G((p\,1,q)\,(2,2))) + T(G(p\,1,q,2,2)).$$

2.157. *Family .(p,q) 1*

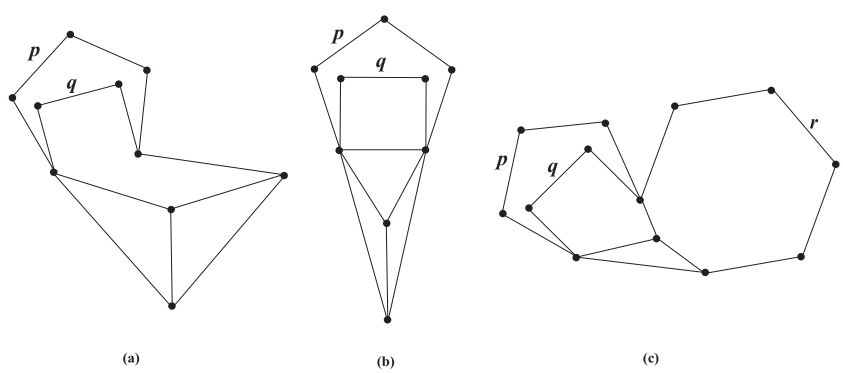


Figure 73: Graphs (a) $G(.(p,q)\,1)$; (b) $G(.(p,q+))$; (c) $G(.(p,q).r)$.

Graph $G(.(p,q)\,1)$ (Fig. 73a) resolves into the graph $G(.(p,q))$ and the block sum of the graphs $G(p+q)$ and G', where G' is the graph which consists of two triangles with a common edge, with the Tutte polynomial $T(G') = x+2x^2+x^3+y+2xy+y^2$. The general formula for the Tutte polynomial of the graphs $G(.(p,q)\,1)$ is

$$T(G(.(p,q)\,1)) = T(G(.(p,q))) + T(G(p+q))(x+2x^2+x^3+y+2xy+y^2).$$

2.158. *Family .(p,q+)*

Graph $G(.(p,q+))$ (Fig. 73b) resolves into graph $G(.(p,q))$ and the block sum of the graphs $G(p)$, $G(q)$, and $\overline{G}(2\,1\,2)$ with the Tutte polynomial $T(\overline{G}(2\,1\,2)) = x+x^2+y+2xy+2y^2+y^3$. The general formula for the Tutte polynomial of the graphs $G(.(p,q+))$ is

$$T(G(.(p,q+))) = T(G(.(p,q))) + T(G(p))T(G(q))(x+x^2+y+2xy+2y^2+y^3).$$

2.159. *Family .(p,q).r*

Graph $G(.(p,q).r)$ (Fig. 73c) resolves into the graphs $G(p,q,r,2)$ and $G'(r\,\overline{2}\,p\,q)$, the block sum of the graphs $G(p+q)$ and $G(r+2)$, and the block sum of the graphs $G(r\,2)$, $G(p)$, and $G(q)$. The general formula for the Tutte polynomial of the graphs $G(.(p,q).r)$ is

$$T(G(.(p,q).r)) = T(G(p,q,r,2)) + T(G'(r\,\overline{2}\,p\,q)) + T(G(p+q))T(G(r+2)) +$$
$$T(G(r\,2))T(G(p))T(G(q)).$$

2.160. *Family .(p,q).r 0*

Graph $G(.(p,q).r\,0)$ (Fig. 74a) resolves into the graphs $G((p,q)\,((r+1),2))$ and $G(p,q,2+r)$, and the block sum of the graphs $G(p,q,2)$ and $G(\overline{r})$. The general formula for the Tutte polynomial of the graphs $G(.(p,q).r\,0)$ is

$$T(G(.(p,q).r\,0)) = T(G((p,q)\,((r+1),2))) + T(G(p,q,2+r)) + T(G(p,q,2))T(G(\overline{r})).$$

2.161. *Family .(p,q):r*

In order to obtain formula for the Tutte polynomial of the graph $G(.(p,q):r)$ (Fig. 74b) we use the relations

$$T(G(.(p,q):r)) - T(G(.(p,(q-1)):r)) = x^{q-1}T(G(.p:r\,0)).$$

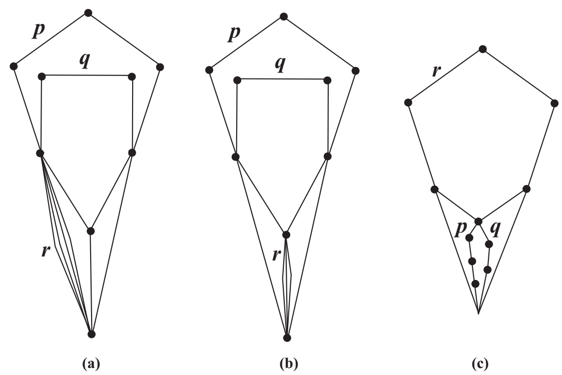


Figure 74: Graphs (a) $G(.(p,q).r\,0)$; (b) $G(.(p,q):r)$; (c) $G(.(p,q):r\,0)$.

Since the Tutte polynomial of the graph $G(.(p,0):r)$ is $T(G(.(p,0):r)) = T(\overline{G}(r,2,2))$, the general formula for the Tutte polynomial of the graphs $G(.(p,q):r)$ is

$$T(G(.(p,q):r)) = \frac{x^q-1}{x-1}T(G(.p:r\,0)) + T(\overline{G}(r,2,2))T(G(p)).$$

2.162. *Family .(p,q):r 0*

In order to obtain formula for the Tutte polynomial of the graph $G(.(p,q):r\,0)$ (Fig. 74c) we use the relations

$$T(G(.(p,q):r\,0)) - T(G(.(p,q):(r-1)\,0)) = x^{r-1}T(G(p,q,2,2)).$$

Since the Tutte polynomial of the graph $G(.(p,q):(0)\,0)$ is $T(G(.(p,q):(0)\,0)) = T(G((p,q)\,(2,2)))$, the general formula for the Tutte polynomial of the graphs $G(.(p,q):r\,0)$ is

$$T(G(.(p,q):r\,0)) = \frac{x^r-1}{x-1}T(G(p,q,2,2)) + T(G((p,q)\,(2,2))).$$

2.163. *Family 8*p 0::q 0*

In order to obtain formula for the Tutte polynomial of the graph $G(8^*p\,0::q\,0)$ (Fig. 75a) we use the relations

$$T(G(8^*p\,0::q\,0)) - T(G(8^*p\,0::(q-1)\,0)) = x^{q-1}T(G(.p:2)).$$

Since the graph $G(8^*p\,0::(0)\,0)$ resolves into the graph $G(p\,2\,1\,2)$ and the block sum of the graphs $G((p+1)\,2)$ and $G(2)$, the general formula for the Tutte polynomial of the graphs $G(8^*p\,0::q\,0)$ is

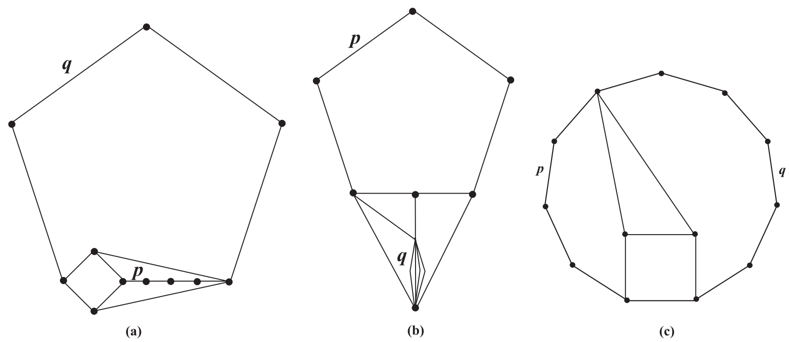


Figure 75: Graphs (a) $G(8^*p\,0 :: q\,0)$; (b) $G(8^*p\,0 : .q\,0)$; (c) $G(8^*p\,0 : q\,0)$.

$$T(G(8^*p\,0 :: q\,0)) = \frac{x^q - 1}{x - 1} T(G(.p : 2)) + T(G(p\,2\,1\,2)) + T(G((p+1)\,2))(x+y).$$

2.164. *Family 8*p 0:.q 0*

Graph $G(8^*p\,0 : .q\,0)$ (Fig. 75b) resolves into the graphs $G(.p.2.q\,0)$ and $G((p, 2+)\,(q, 1))$, and the block sum of the graphs $G(2\,q)$ and $G(p+2)$. The general formula for the Tutte polynomial of the graphs $G(8^*p\,0 : .q\,0)$ is

$$T(G(8^*p\,0 : .q\,0)) = T(G(.p.2.q\,0)) + T(G((p, 2+)\,(q, 1))) + T(G(2\,q))T(G(p+2)).$$

2.165. *Family 8*p 0:q 0*

Graph $G(8^*p\,0 : q\,0)$ (Fig. 75c) resolves into the graphs $G(.(p, q))$ and $G((p+1)\,1\,1\,1\,(q+1))$. The general formula for the Tutte polynomial of the graphs $G(8^*p\,0 : q\,0)$ is

$$T(G(8^*p\,0 : q\,0)) = T(G(.(p, q))) + T(G((p+1)\,1\,1\,1\,(q+1))).$$

2.166. *Family 8*p 0.q 0*

Graph $G(8^*p\,0.q\,0)$ (Fig. 76a) resolves into the graphs $G(.1.q.p.2)$ and $G((2\,1, q)\,(p, 2))$. The general formula for the Tutte polynomial of the graphs $G(8^*p\,0.q\,0)$ is

$$T(G(8^*p\,0.q\,0)) = T(G(.1.q.p.2)) + T(G((2\,1, q)\,(p, 2))).$$

2.167. *Family 8*p 1*

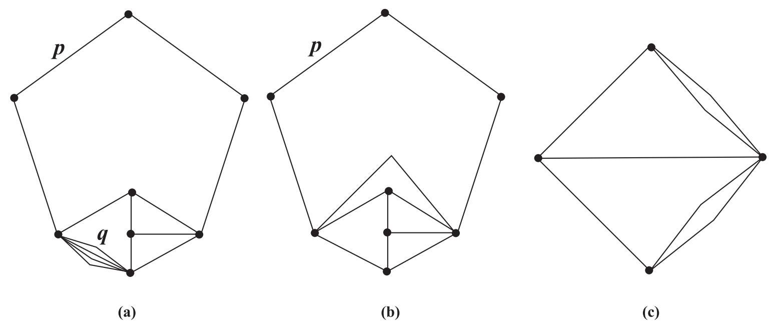


Figure 76: Graphs (a) $G(8^*p\,0.q\,0)$; (b) $G(8^*p\,1)$; (c) G'.

Graph $G(8^*p\,1)$ (Fig. 76b) resolves into the graph $G(8^*p\,0)$ and the block sum of the graphs $G(p)$ and G' (Fig. 76c) with the Tutte polynomial $T(G') = x+2x^2+x^3+y+4xy+2x^2y+3y^2+3xy^2+3y^3+y^4$. The general formula for the Tutte polynomial of the graphs $G(8^*p\,1)$ is

$$T(G(8^*p\,1)) = T(G(8^*p\,0))+T(G(p))(x+2x^2+x^3+y+4xy+2x^2y+3y^2+3xy^2+3y^3+y^4).$$

2.168. *Family 8* p 1 0*

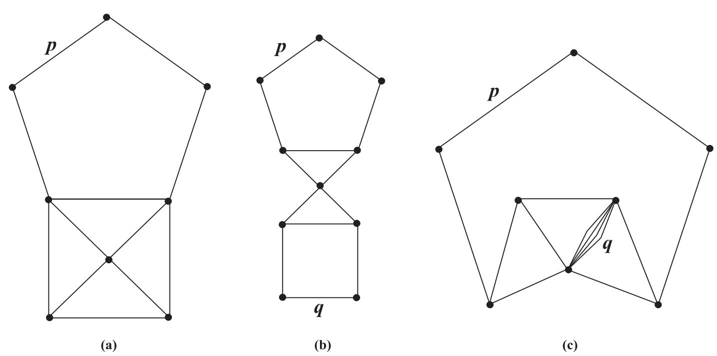


Figure 77: Graphs (a) $G(8^*p\,1\,0)$; (b) $G(8^*p :: q)$; (c) $G(8^*p : .q)$.

Graph $G(8^*p\,1\,0)$ (Fig. 77a) resolves into the graph $G(8^*p)$ and the block sum of the graphs $G(p)$ and $G(.1 : 2\,0)$ with the Tutte polynomial $T(G(.1 : 2\,0)) = 2x+3x^2+x^3+2y+5xy+x^2y+4y^2+2xy^2+3y^3+y^4)$. The general formula for the Tutte polynomial of the graphs $G(8^*p\,1\,0)$ is

$T(G(8^*p\,1\,0)) = T(G(8^*p)) + T(G(p))(2x+3x^2+x^3+2y+5xy+x^2y+4y^2+2xy^2+3y^3+y^4)$.

2.169. *Family 8*p::q*

Graph $G(8^*p :: q)$ (Fig. 77b) resolves into the graphs $G((p+1)\,1\,1\,1\,(q+1))$ and $G(1\,0.p.q\,0.2\,0)$. The general formula for the Tutte polynomial of the graphs $G(8^*p :: q)$ is

$$T(G(8^*p :: q)) = T(G((p+1)\,1\,1\,1\,(q+1))) + T(G(1\,0.p.q\,0.2\,0)).$$

2.170. *Family 8*p:.q*

Graph $G(8^*p : .q)$ (Fig. 77c) resolves into the graphs $G((p+1)\,1\,1\,1\,1\,q)$ and $G(1\,0.p.1\,0.(q+1)\,0)$. The general formula for the Tutte polynomial of the graphs $G(8^*p : .q)$ is

$$T(G(8^*p : .q)) = T(G((p+1)\,1\,1\,1\,1\,q)) + T(G(1\,0.p.1\,0.(q+1)\,0)).$$

2.171. *Family 8*p:q*

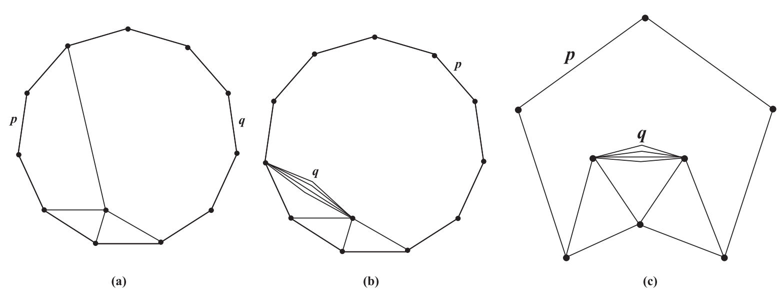


Figure 78: Graphs (a) $G(8^*p : q)$; (b) $G(8^*p.q)$; (c) $G(8^*p :: q\,0)$.

Graph $G(8^*p : q)$ (Fig. 78a) resolves into the graphs $G(.(p+q))$ and $G(p\,1\,2\,1\,q)$, and the block sum of the graphs $G(p\,2)$ and $G(q\,2)$. The general formula for the Tutte polynomial of the graphs $G(8^*p : q)$ is

$$T(G(8^*p : q)) = T(G(.(p+q))) + T(G(p\,1\,2\,1\,q)) + T(G(p\,2))T(G(q\,2)).$$

2.172. *Family 8*p.q*

Graph $G(8^*p.q)$ (Fig. 78b) resolves into the graphs $G(.p.q.1\,0)$ and $G(p\,(\overline{q+1})\,\overline{3})$, the block sum of the graphs $G((p+1)\,(q+1))$ and $G(2)$, and the block sum of the graphs $G(2\,1,q,1+(p-1))$ and $G(1)$. The general formula for the Tutte polynomial of the graphs $G(8^*p.q)$ is

$$T(G(8^*p.q)) = T(G(.p.q.1\,0)) + T(G(p\,(\overline{q+1})\,\overline{3})) + T(G((p+1)\,(q+1)))(x+y)+$$

$$xT(G(2\,1,q,1+(p-1))).$$

2.173. *Family 8*p::q 0*

Graph $G(8^*p :: q\,0)$ (Fig. 78c) resolves into the graphs $G(.(p+1).q)$ and $G(p\,1\,1\,1,q,2)$. The general formula for the Tutte polynomial of the graphs $G(8^*p :: q\,0)$ is

$$T(G(8^*p :: q\,0)) = T(G(.(p+1).q)) + T(G(p\,1\,1\,1,q,2)).$$

2.174. *Family 8*p:.q 0*

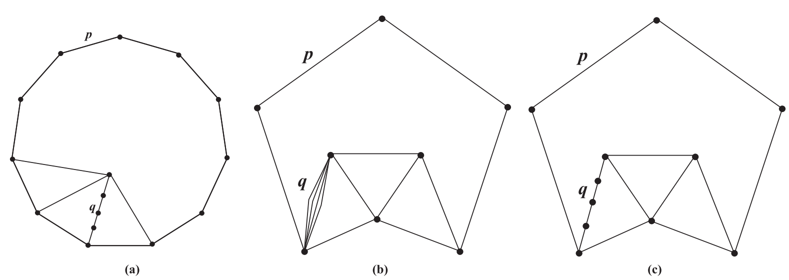


Figure 79: Graphs (a) $G(8^*p : .q\,0)$; (b) $G(8^*p : q\,0)$; (c) $G(8^*p.q\,0)$.

Graph $G(8^*p : .q\,0)$ (Fig. 79a) resolves into the graphs $G(.q\,1 : p\,0)$ and $G((q+1)\,1\,p\,1\,2)$. The general formula for the Tutte polynomial of the graphs $G(8^*p : .q\,0)$ is

$$T(G(8^*p : .q\,0)) = T(G(.q\,1 : p\,0)) + T(G((q+1)\,1\,p\,1\,2)).$$

2.175. *Family 8*p:q 0*

Graph $G(8^*p : q\,0)$ (Fig. 79b) resolves into the graphs $G(p\,1\,1\,1\,1\,(q+1))$, $G(p\,\overline{q}\,\overline{2}\,\overline{2})$ and $G'(p\,\overline{q}\,2\,2)$. The general formula for the Tutte polynomial of the graphs $G(8^*p : q\,0)$ is

$$T(G(8^*p:q\,0)) = T(G(p\,1\,1\,1\,1\,(q+1))) + T(G(p\,\overline{q}\,\overline{2}\,\overline{2})) + T(G'(p\,\overline{q}\,2\,2)).$$

2.176. *Family 8*p.q 0*

In order to obtain formula for the Tutte polynomial of the graph $G(8^*p.q\,0)$ (Fig. 79c) we use the relations

$$T(G(8^*p.q\,0)) - T(G(8^*p.(q-1)\,0)) = x^{q-1}T(G(.(p+1))).$$

Since the Tutte polynomial of the graph $G(8^*p.(0)\,0)$ is $T(G(8^*p.(0)\,0)) = T(G(p\,1\,1\,1\,1\,2))$, the general formula for the Tutte polynomial of the graphs $G(8^*p.q\,0)$ is

$$T(G(8^*p.q\,0)) = \frac{x^q-1}{x-1}T(G(.(p+1))) + T(G(p\,1\,1\,1\,1\,2)).$$

2.177. *Family 9*.p*

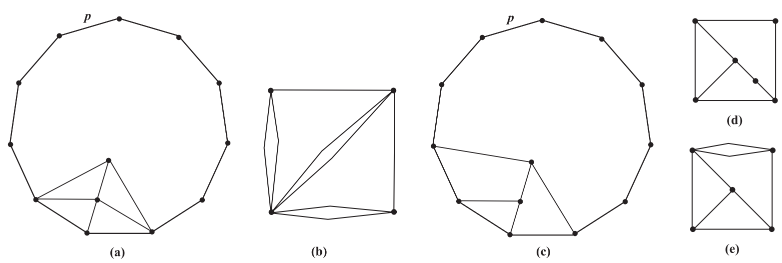


Figure 80: Graphs (a) $G(9^*.p)$; (b) G'; (c) $G(9^*p)$; (d) G''; (e) G'''.

In order to obtain formula for the Tutte polynomial of the graph $G(9^*.p)$ (Fig.80a) we use the relations

$$T(G(9^*.p)) - T(G(9^*.(p-1))) = x^{p-1}T(Wh(5)).$$

where $Wh(5)$ is the wheel graph with the Tutte polynomial

$$T(Wh(5)) = 3x + 6x^2 + 4x^3 + x^4 + 3y + 9xy + 4x^2y + 6y^2 + 4xy^2 + 4y^3 + y^4.$$

Since $G' = G(9^*.(0))$ is the graph G' (Fig. 80b) with the Tutte polynomial

$$T(G') = x + 2x^2 + x^3 + y + 4xy + 3x^2y + 3y^2 + 5xy^2 + 4y^3 + 2xy^3 + 3y^4 + y^5,$$

the general formula for the Tutte polynomial of the graphs $G(9^*.p)$ is

$$T(G(9^*.p)) = \frac{x^p-1}{x-1}T(Wh(5)) + T(G').$$

2.178. *Family 9* p*

In order to obtain formula for the Tutte polynomial of the graph $G(9^*p)$ (Fig. 80c) we use the relations

$$T(G(9^*p)) - T(G(9^*(p-1))) = x^{p-1}T(G'),$$

where G'' is the graph (Fig. 80d) with the Tutte polynomial

$$T(G') = 2x + 5x^2 + 5x^3 + 3x^4 + x^5 + 2y + 6xy + 4x^2y + x^3y + 3y^2 + 2xy^2 + y^3.$$

Since $G''' = G(9^*(0))$ is the graph (Fig. 80e) with the Tutte polynomial

$$T(G''') = 2x + 4x^2 + 3x^3 + x^4 + 2y + 7xy + 5x^2y + x^3y + 5y^2 + 5xy^2 + 4y^3 + y^4,$$

the general formula for the Tutte polynomial of the graphs $G(9^*p)$ is

$$T(G(9^*p)) = \frac{x^p - 1}{x - 1}T(G'') + T(G''').$$

2.179. *Family 9*.p 0*

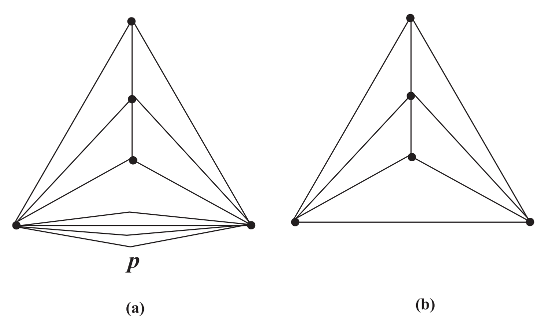


Figure 81: Graphs (a) $G(9^*.p\,0)$; (b) G''.

In order to obtain formula for the Tutte polynomial of the graph $G(9^*.p\,0)$ (Fig. 81a) we use the relations

$$T(G(9^*.p\,0)) - T(G(9^*(p-1)\,0)) = y^{p-1}T(G'),$$

where G' is the graph with the Tutte polynomial

$$T(G') = x + 2x^2 + x^3 + y + 4xy + 3x^2y + 3y^2 + 5xy^2 + 4y^3 + 2xy^3 + 3y^4 + y^5.$$

Since $G'' = 9^*.(1)$ is the graph G'' (Fig. 81b) with the Tutte polynomial

$$T(G'') = 4x + 8x^2 + 5x^3 + x^4 + 4y + 13xy + 7x^2y + 9y^2 + 9xy^2 + 8y^3 + 2xy^3 + 4y^4 + y^5,$$

the general formula for the Tutte polynomial of the graphs $G(9^*.p\,0)$ is

$$T(G(9^*.p\,0)) = \frac{y^{p-1}-1}{y-1} yT(G') + T(G'').$$

3. Conclusion

In conclusion, we provide explicit formulae for the Tutte polynomials of all families of KLs derived from source links with at most 10 crossings. The Jones polynomials of all alternating and non-alternating knots and links which belong to the families derived from source links up to 10 crossings can be obtained by substituting $x \to -x$ and $y \to -\frac{1}{x}$ [Bo, ChaShro].

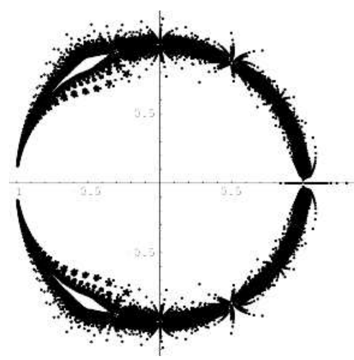


Figure 82: Plot of zeros of Jones polynomials for the family $p\,q$ ($2 \leq p \leq 50$, $2 \leq q \leq 50$).

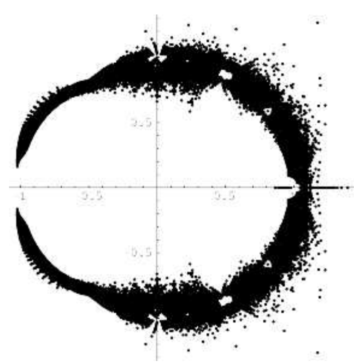


Figure 83: Plot of zeros of Jones polynomials for the family of alternating pretzel KLs p, q, r ($2 \leq p \leq 20$, $2 \leq q \leq 20$, $2 \leq r \leq 20$).

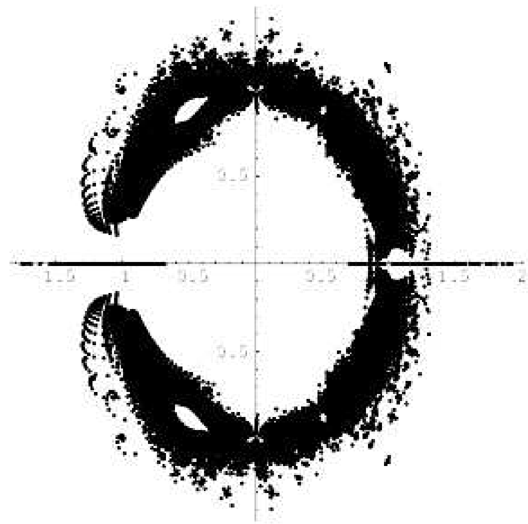


Figure 84: Plot of zeros of Jones polynomials for the family of non-alternating pretzel KLs $p, q, -r$ ($2 \leq p \leq 20$, $2 \leq q \leq 20$, $2 \leq r \leq 20$).

Obtained results can be used for studying the Tutte and Jones polynomials of KL families. For example, the leading coefficient of the Tutte polynomial is equal to 1 for all alternating algebraic KLs, and greater then 1 for non-algebraic ones. In particular, it is invariant for the Tutte polynomials of graphs corresponding to knots and links inside a family derived from a basic polyhedron. Additionally they can be used for studying zeros of the Jones polynomials for the whole family. For example, we plot all zeroes of the Jones polynomial [ChaShro, JiZha, WuWa] for KL family, referred to as the characteristic "portrait of family".

The plot of zeroes of Jones polynomials for the alternating link family $p\,q$ ($2 \leq p \leq 50$, $2 \leq q \leq 50$) is shown in Fig. 82, the plot of zeroes of Jones polynomials for the alternating link family p, q, r ($2 \leq p \leq 20$, $2 \leq q \leq 20$, $2 \leq r \leq 20$) is shown in Fig. 83, and the plot of zeroes of Jones polynomials for the non-alternating link family $p, q, -r$ ($2 \leq p \leq 20$, $2 \leq q \leq 20$, $2 \leq r \leq 20$) is shown in Fig. 84. More detailed results of this kind will be given in the forthcoming paper.

The paper is accompanied with the *Mathematica* package downloadable from the address

`http://www.mi.sanu.ac.rs/vismath/Tutte.htm`

which provides readers with the possibility to compute all the results for the given KL families. In particular, the function **GraphFam** computes the graph corresponding to a KL. Functions **TutteFam** and **JonesFam** compute Tutte and Jones polynomials of the graphs or links, respectively. Moreover, there are three functions which provide additional information about zeros of the Jones polynomial: **Zeros** computes zeros of a particular Jones polynomial, **ZeroSum** outputs

the sum of their absolute values, and **Portrait** produces the plot of zeros of the Jones polynomials of a given family. All functions are based on general formulas, so there are no restrictions on the number of crossings of KLs except for hardware ones.

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